Cognition 189 (2019) 89–104

Contents lists available at ScienceDirect

Cognition

journal homepage: www.elsevier.com/locate/cognit

Preschoolers and multi-digit numbers: A path to mathematics through the symbols themselves☆

Lei Yuan[a,*], Richard W. Prather[b], Kelly S. Mix[b], Linda B. Smith[a,c]

[a] Indiana University, United States
[b] University of Maryland, United States
[c] School of Psychology, University of East Anglia, Norwich, Norfolk, UK

ARTICLE INFO

Keywords:
Symbolic numbers
Non-symbolic quantities
Early learning
Relational learning
Education

ABSTRACT

Numerous studies from developmental psychology have suggested that human symbolic representation of numbers is built upon the evolutionally old capacity for representing quantities that is shared with other species. Substantial research from mathematics education also supports the idea that mathematical concepts are best learned through their corresponding physical representations. We argue for an independent pathway to learning "big" multi-digit symbolic numbers that focuses on the symbol system itself. Across five experiments using both between- and within-subject designs, we asked preschoolers to identify written multi-digit numbers with their spoken names in a two-alternative-choice-test or to indicate the larger quantity between two written numbers. Results showed that preschoolers could reliably map spoken number names to written forms and compare the magnitudes of two written multi-digit numbers. Importantly, these abilities were not related to their non-symbolic representation of quantities. These findings have important implications for numerical cognition, symbolic development, teaching, and education.

1. Introduction

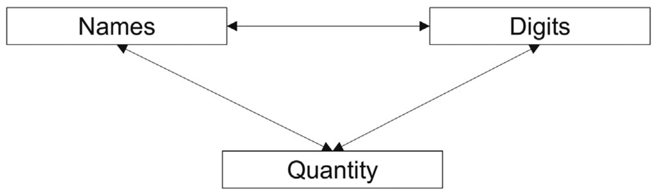
Human knowledge about numbers has been characterized as a triple-code system (Dehaene, 1992; Dehaene & Cohen, 1995): discrete quantities can be represented by sets of the physical quantities themselves, by number names, or by written symbols. There are many reasons to believe that the perception of physical quantities is the starting point historically and developmentally. Consistent with this idea, many studies using various methods have shown strong predictive relations between the perceptual discrimination of quantities and mathematics achievement (e.g., Barth et al., 2006; Dehaene, 2011; Gallistel & Gelman, 1992; Gilmore, McCarthy, & Spelke, 2010; Libertus, Feigenson, & Halberda, 2013; Piazza et al., 2010). These include correlational studies linking children's perceptual discriminations with current and later mathematics performance (Bonny & Lourenco, 2013; Chen & Li, 2014; Feigenson, Libertus, & Halberda, 2013; Gilmore et al., 2010; Halberda, Mazzocco, & Feigenson, 2008; Inglis, Attridge, Batchelor, & Gilmore, 2011), analyses of the perceptual discrimination skills of children with mathematics disabilities (Mazzocco, Feigenson, & Halberda, 2011; Piazza et al., 2010), as well as demonstrations that perceptual training benefits performance in mathematics tasks (Hyde, Khanum, & Spelke, 2014; Park & Brannon, 2013; Räsänen, Salminen, Wilson, Aunio, & Dehaene, 2009; Wilson, Dehaene, Dubois, & Fayol, 2009). Given these facts, it seems plausible that the path to understanding the other two parts of the triple-code system—number names and written symbols—is through the perception of physical quantities (Barth, Starr, & Sullivan, 2009; Park & Brannon, 2013; Piazza, Izard, Pinel, Le Bihan, & Dehaene, 2004). Indeed, in his influential book, Dehaene (1992) suggested that number names were directly converted to meaning through perceptual representations of quantity.

Here, we ask: Could learning about the symbol system, specifically with respect to big numbers, offer a second independent entrance to mathematics? Mathematics at its core is not about specific quantities but rather is about the systems of relations among quantities as variables (Russell, 1903). The notational and naming system we use to represent specific quantities are founded on a system of relations, representing and naming quantities as counts within a multiplicative hierarchy of sets of 10 (Copi, Menninger, & Broneer, 1971). Thus, the symbol "342" is named "three-hundred and forty-two" and counts three sets of one hundred, four sets of ten, and two sets of one, with 10 sets of

☆ This research was supported by NSF DRL 1621-93 grant to Linda Smith and Kelly Mix, IES R305A080287 grant to Linda Smith and Kelly Mix, and NICHD F32 HD090827-02 grant to Lei Yuan.
* Corresponding author.
E-mail address: leiyuan@indiana.edu (L. Yuan).

https://doi.org/10.1016/j.cognition.2019.03.013
Received 11 September 2018; Received in revised form 15 March 2019; Accepted 21 March 2019
0010-0277/

A. The triple-code theory of number representation



B. Experimental design and the formats being tested in each experiment

Task	Which-N Task		Which-More Task		Which-N & Which-More
Experiment	E1	E2	E3	E4	E5
Design	Between-s	Within-s	Between-s	Within-s	Within-s
Stimuli	*"Which is twenty-six?"* Symbol condition: 26 206; Dot condition		*"Which is more?"* Symbol condition: 307 37; Dot condition		*"Which is twenty-six?"* 206 26; *"Which is more?"* 37 307
Formats being tested	Name → Digit Name → Quantity		Digit Quantity		Name → Digit Digit

Fig. 1. (A) A visual illustration of the triple-code theory of number representation. Numbers can be represented with three formats—names, digits or quantity. These formats can be directly converted to each other. B) Overview of the five experiments. Experiment 1 used a between-subject design and Experiment 2 used a within-subject design to compare children's ability to (1) map names for large (2 and 3 digits) numbers to written digits and (2) to map names to dot arrays. The formats being tested are the conversion from names to digits and the conversion from names to quantities respectively. Experiment 3 and 4 used between- and within-subject designs to examine children's ability to compare (1) the relative magnitudes of large quantities given written multi-digit numbers or (2) the relative magnitudes of large quantities given dot arrays of those same quantities. The formats being tested are digits and quantities. Experiment 5 provides evidence on the link between children's ability to (1) map number names to written digits and (2) their ability to make magnitude judgments given written representations of quantities. The formats being tested are the conversion from names to digits and digits alone.

one equal to 1 set of ten, and 10 sets of ten equal to 100 sets of one. Likewise, the symbol "546" is named "five-hundred and forty-six" and counts 5 sets of one hundred, 4 sets of ten and 6 sets of one. Although 342 and 546 refer to different specific quantities, the written forms and spoken names reflect the same relational structure. Computational models have shown that learners could, in principle, capitalize on these regularities within and across spoken and written names to learn the underlying relational structure—enabling such a learner to understand never-before-seen multi-digit numbers—without any grounding of the represented number to a specific perceived quantity (Grossberg & Repin, 2003; Rule, Dechter, & Tenenbaum, 2015). These kinds of models derive the underlying structure that maps spoken numbers to written numbers and thus demonstrate that learning about the surface properties of the notational system and their names could be a path into understanding place value notation and as an introduction to mathematics as a relational system.

However, the consensus view from research on children's learning of the place value concepts provides little support for this idea. The difficulty of place value concepts for children—evident late into elementary school—is well-documented (Ross, 1986; Fuson & Briars, 1990; Fuson, 1990; Gervasoni et al., 2011; Ross and Sunflower, 1995). The irregularities in number names that characterize many languages (e.g., "eleven", "fifteen" in English) are known to cause children considerable difficulty and have led some researchers to conclude that children cannot discover place value principles from the surface structure of names and written numbers alone (Fuson & Kwon, 1992; Miura & Okamoto, 1989; Saxton & Towse, 1998). Further, many theorists have argued that curricula designed to ground place value notation in discrete counts and physical models benefit learning about place value, although there is mixed evidence in support of this conclusion (Mix, 2010; Mix, Smith, & Crespo, 2019; Mix, Smith, Stockton, Cheng, & Barterian, 2017). Recently, researchers have further suggested that large number meanings might be grounded in approximate perceptual representations of ungrouped quantities (Barth et al., 2009; Piazza et al., 2004), although recent evidence suggests that these links may not be easily formed by children (Sullivan & Barner, 2011). All this would seem to suggest that understanding the place value system is difficult, not easily linked to perceived quantities, and thus likely to require explicit instruction.

These conclusions may be missing an important role for early informal learning about multi-digit numbers and their names, the kind of early learning that we hypothesize does not involve linking large number names or their written forms to perceived quantities. This hypothesis is suggested by several findings showing that preschool children know more about multi-digit numbers than one would expect given the difficulties of school-age children (Byrge, Smith, & Mix, 2014; Mix, Prather, Smith, & Stockton, 2014). One study (Mix et al., 2014) presented 4- and 5-year-old children with 2-, 3- and 4-place multi-digit numbers in a 2-alternative-forced-choice task and asked them to indicate the one that matched a spoken number name ("Which is N?") or was of greater magnitude ("Which is more?"). The children performed well above chance. These preschool children's performances certainly do not implicate an explicit understanding of base-10 notation in the sense of knowing that the 6 in 642 is 6 sets of 100, that the 4 is 4 sets of 10, and that the 2 is 2 sets of one—the goal of formal training about place value in school-age children. However, the likelihood that these children had encountered the name and written form of any individual 3- or 4-digit numbers used in the study (e.g., 836) is vanishingly small given the sparsity of individual number names in talk to preschool children (Dehaene, 1992; Dehaene & Mehler, 1992; Levine, Suriyakham, Rowe, Huttenlocher, & Gunderson, 2010). Yet these children showed implicit knowledge about how (likely unfamiliar) number names and written forms work. Evidently, preschool children, without formal instruction, are deriving generalizable knowledge about the two symbolic codes in the triple-code system.

The five experiments that follow were designed to further document this emerging knowledge and to test the hypothesis that this early understanding of the symbolic code is not dependent on children's ability to map specific multi-digit number names to the physical quantities they represent. As illustrated in Fig. 1, our approach was to probe the connections between the three codes in Dehaene (1992) triple-code system. In Experiments 1 and 2, we directly assessed two key links in the triple-code system: number names to written digits and number names to physical quantities. Experiment 1 used a between-subject cross-sectional design with the explicit purpose of collecting data from a large and diverse sample of preschool children. Prior studies (Byrge et al., 2014; Mix et al., 2014) suggesting early knowledge about the place value system were based on small samples potentially drawn from selective populations thus limiting general conclusions about this early knowledge about multi-digit number names and written forms. Experiment 2 used a smaller sample within-subject design to assess the relation between these two links—names to digits and names to quantities. Experiments 3 and 4 (in a large-N between-subjects study and in a smaller N within-subject study, respectively) examined children's abilities to make relative magnitude judgements about multi-digit numbers and the physical quantities represented by those numbers. Considerable past research shows that children are quite skilled in making relative magnitude judgements about physical quantities (Halberda & Feigenson, 2008; Libertus, Feigenson, & Halberda, 2011; Odic, Libertus, Feigenson, & Halberda, 2012): Are the developmental trajectories for quantities and symbols similar? Are individual children's abilities to compare specific written digits predicted by their comparisons of the same physical quantities? The answers should be "yes" if the perception of physical quantities provides the meaning for big numbers and their written representations.

Experiment 5 focused on children's understanding of the two symbolic code—names and written forms—and whether children's ability to make relative magnitude judgments about written digits is predicted by their ability to link number names to those written forms. Finally, in a sixth section of the Results, we aggregated the data involving digits and perceived quantities across the five experiments, providing a finer characterization of growth in knowledge about written symbols and the physical quantities in children from 3 to 6 years of age.

Altogether the results provide three kinds of evidence pertinent to the hypothesis that young children are building a beginning understanding of the notational system that is not dependent on their ability to make judgements about the physical quantities being represented: the results (1) provide an assessment of early knowledge about the names and written forms of multi-digit numbers in a broad sample of children; (2) directly compare the patterns of growth in children's judgments of number names, written digits, and physical quantities; and (3) examine the within-subject relation between judgments of the multi-digit numbers and the physical quantities they represent.

2. Experiment 1

Experiment 1 compared children's ability to map heard number names to written forms or to dot arrays. For this study, we purposefully focused on test items that should be difficult in the case of mapping names to digits, but easy in the case of mapping names to physical quantities. Research on developing place value knowledge suggests that written forms with zeros are challenging for young learners (Byrge et al., 2014; Zuber, Pixner, Moeller, & Nuerk, 2009). When a novice learner is presented with the name "twenty one" and asked to choose between the written forms "21" and "201," both choices include a 2 and a 1, and if children do not recognize the role of places, "201" could be construed as having a "20" and "1." However, the physical quantities represented by 21 and 201 differ by over 9-fold. If children understand number names in terms of (approximate) representations of the physical quantity, then mapping "twenty one" to a set of 21 things rather than to a set of 201 things should be relatively easy. Because young children are more likely to have stronger and more precise representations of small set sizes than larger ones (Feigenson, Dehaene, & Spelke, 2004; Prather, 2012) and also consistent with the use of this task in assessments of school-age children's understanding of multi-digit numbers (Mix et al., 2014), the presented name in both the Dots and Digits conditions always asked for the smaller quantity (e.g., "twenty-one" in a comparison of "21" vs "201").

2.1. Method

Participants. The participants were 176 children from 3 to 6 years of age, with half male and half female in each of 4 broad age groups: 44 3-year-olds (mean = 43.4 mo, range = 38 to 47 mo); 43 4-year-olds (mean = 54.2, range = 48 to 59 mo); 45 5-year-olds (mean = 65.3 mo, range 60 to 71 mo), and 44 6-year-olds (mean = 79.4 mo, range = 72 to 84 mo). The sample of children was broadly representative of the local population: 84% European American, 5% African American, 5% Asian American, 2% Latino, 4% Other) and consisted of predominantly working- and middle-class families. Children were recruited through community organizations (e.g., museums, child outreach events, boys' and girls' clubs) and at 12 different preschools and daycares selected to serve a diverse income population. Eighteen percent of the children attended daycares or lived in neighborhoods serving schools with over a 50% participation in the free-lunch program. Most of the 5- and 6-year-olds were in some form of half-day kindergarten (at a public school or in daycare); kindergarten is not required by the local state and the curriculum varies considerably across different schools. Counting to 100 and exposure to the corresponding written digits were part of some children's kindergarten experiences. Children were assigned to either the Digits Condition or the Dots Conditions; equal numbers of children from each participating school or school district were assigned to the two conditions.

Stimuli. To accommodate the goal of testing a broad sample of young children in a variety of contexts, each child was tested on only 10 two-alternative-forced-choice trials: 3 v 7, 11 v 24, 15 v 105, 21 v 201, 36 v 306, 42 v 402, 64 v 604, 78 v 807, 206 v 260 and 305 v 350. Six of the trials compared two and three-digit numbers that differed by the presence of a zero. In addition to these 6 items, we included one single digit comparison, one two-digit comparison, and two 3-digit comparisons in which the choices were transpositions and included a zero. These last two items were expected to be difficult in both the Digits and Dots conditions.

For the Dots condition, the comparison sets were arrays of happy-face dots as shown in Fig. 2. The to-be-compared arrays for each trial were presented on an 11-inch by 8-inch card with each array centered in its half of the card. The dots were randomly placed within an irregular region of the same area such that the density (or inter-dot distance) co-varied with set size while the overall area was comparable. Dot arrays that vary in quantity always co-vary with other properties (e.g., cumulative area, overall area, density, see Cantrell & Smith, 2013). Studies directed to measuring children's *precision* of discriminating dot arrays often use many trials and kinds of arrays to attempt to control for children's possible dependence on these other co-varying dimensions (Leibovich & Henik, 2013; Odic et al., 2012). In the present study, the arrays roughly equate the area of the to-be-compared arrays but do not control for other possible dimensions. We chose this approach on two grounds: First, dot arrays that differ in the number of dots, like perceived quantities in the world, co-vary across a number of dimensions and it has been argued that these arrays provide the clearest signal to the physical quantities of the arrays (Cantrell, Boyer, Cordes, & Smith, 2015; Cantrell & Smith, 2013). Second, the question under test is not that the *precision* of discrete quantity comparison predicts learning about visual symbols, but that children's learning about number names and multi-digit numbers is or is not dependent on their ability to make the same judgements with respect to physical quantities. Accordingly, two unique sets of arrays were constructed for each trial that differed in the random arrangement of the dots and the side of the correct choice.

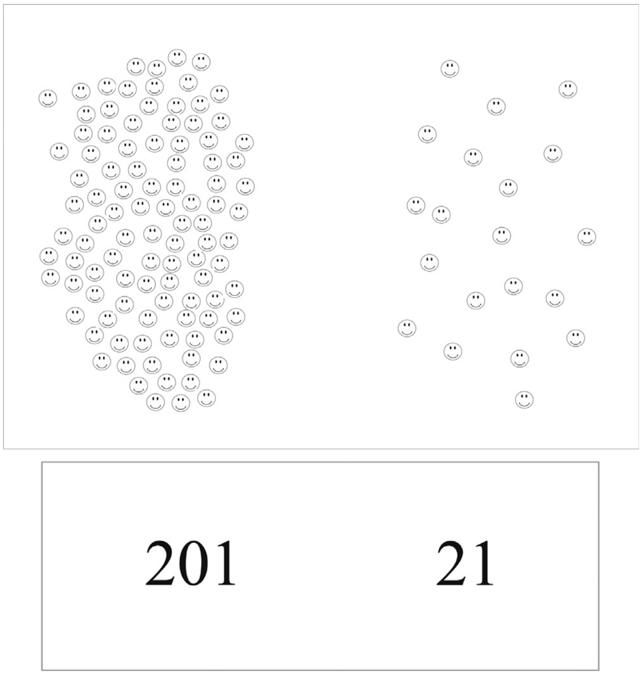


Fig. 2. Top: sample stimuli from the Dots condition, depicting the 201 (left) and 21 (right) smiley faces. Bottom: sample stimuli from the Digits condition.

Half the children in each age group in the Dots condition were presented one of these two sets of arrays.

For the Digits condition, the digits were printed in 100 pt font (Times New Roman) and presented on 8.5-inch × 3-inch cards with each number centered in its half of the card as shown in Fig. 2. Two versions of these cards were also constructed with the left-right position of the to-be-compared numbers counterbalanced across the two sets and with half the children in each age group in the Digits condition receiving one of these sets.

In both conditions, for all card sets, half of the correct choices were on the left and half were on the right and the order of test trials was randomly determined for each subject. All cards were laminated in plastic.

Procedure. The children were tested in a quiet room. There were two warm-up trials using cards that showed two objects (i.e., dog, cup) located on separate sides of the cards. The experimenter asked the child to indicate the named object, saying "Look at these. Look at this one. Now, look at this one. Look at them both before you make your choice. Which one is ___?" Children were asked to indicate by pointing to the labeled side. All children correctly did so on the two warm-up trials. For the immediately following 10 test trials, the experimenter said: "now I am going to say a number and I want you to tell me which picture shows that number." She presented the two choices and said "Look at this one. Now, look at this one. Look at them both before you make your choice. Which one is ___?" The experimenter offered no feedback of any kind as she proceeded through the 10 trials. Pilot studies indicated a bias among very young children to always choose the larger dot array. Accordingly, to be conservative in measuring children's ability to map number names to dots, children were asked—in both the Digits and the Dots conditions—for the smaller of the two numbers or amounts.

Analysis Plan. The goal of the between-subject experiments (1 and 3) is to compare developmental differences and group differences in the Digits and Dots conditions. To this end we use three converging approaches in the analysis: (1) To measure the incremental growth of judgements of the two kinds, we used regression analyses relating individual children's performance in each task to continuous age; (2) to directly compare performance in the two conditions as function of age, we grouped the children into 4 age groups and used a 4 (Age group) by 2 (Condition) analysis of variance comparing overall performance; and

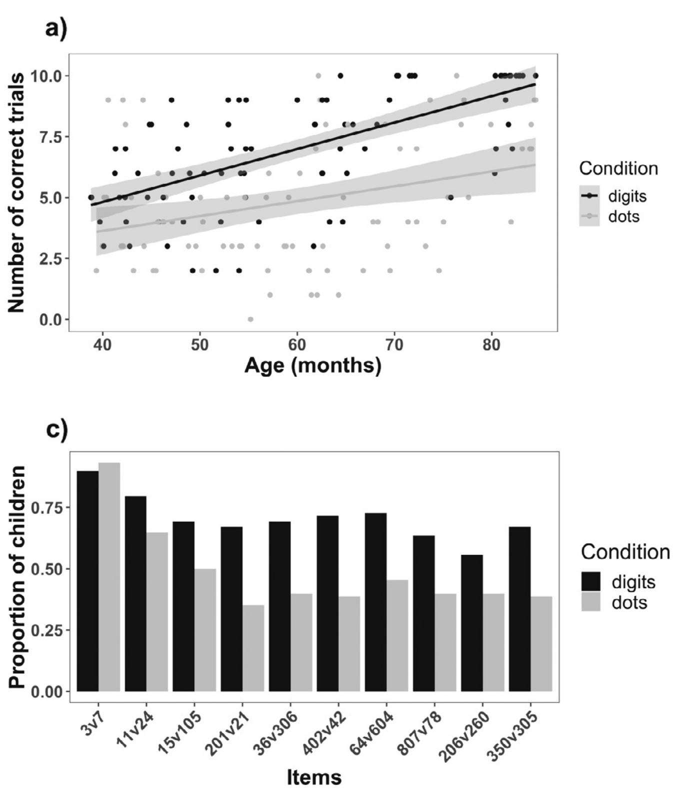


Fig. 3. Results from Experiment 1: (a) The number of trials with the correct answer for the Digits and Dots tasks as a function of age. (b) The average number of correct trials for the Digits and Dots task in each age group. The dotted line indicates chance performance. (c) The proportion of children who answered correctly for each of the test items in the Digits and Dots tasks.

(3) to examine item-level effects, we used a Chi-square analyses of the number of children (collapsed across age group) answering each item type correctly in the two conditions. Because gender differences in mathematics abilities are typically not observed in preschool children (Geary, 1994; Jordan, Kaplan, Ramineni, & Locuniak, 2009; Lachance & Mazzocco, 2006; Lummis & Stevenson, 1990) but have been reported in a few studies (Byrge et al., 2014; Ginsburg & Russell, 1981; Robinson, Abbott, Berninger, & Busse, 1996), in all experiments we applied a highly sensitive measure of possible gender differences, comparing the male versus female performance for all children within each condition by a simple *t*-test. Across the 5 experiments, we observed no gender differences, $ps > 0.50$ for all comparisons. We do not consider this factor further.

2.2. Results and discussion

Fig. 3a shows each child's number of correct trials as a function of continuous age in the two conditions. Performance increased consistently with age in the Dots condition, $R^2 = 0.08$, $F(1, 86) = 9.01$, $p = .003$, and in the Digits condition, $R^2 = 0.43$, $F(1, 86) = 66$, $p < .0001$. As is also clear, many children, including some quite young children, performed very well in the Digits version of the task whereas many children, including many older children, performed below chance in the Dots task. As is also apparent, age-related growth was steeper in the Digits than Dots task ($B = 0.66$ and 0.31 respectively). Fig. 3b shows the mean performances in the Dots and Digits conditions as a function of the four age groups. A 4 (Age group) by 2 (Stimulus condition) analysis of variance yielded only reliable main effects of Stimulus condition, $F(1, 168) = 46.4$, $p < .0001$, partial $\eta^2 = 0.22$, and Age group, $F(3, 168) = 19.58$, $p < .0001$, partial $\eta^2 = 0.26$. The interaction between Stimulus condition and Age group approached conventional significance, $F(3, 168) = 2.67$, $p = .049$, partial $\eta^2 = 0.04$. As shown in Fig. 3b, as a group, 5- and 6-year old children performed very well in mapping number names to Digits, and did so at levels greater than chance, $t_{5yearolds}(21) = 6$, $p < .0001$; $t_{6yearolds}(21) = 13$, $p < .0001$, two-tailed. In contrast, in the Dots task, only the 6-year-olds performed above chance ($t(21) = 3.2$, $p = .005$), all other age groups performed at chance level or reliably below chance, $t_{3yearolds}(21) = -0.54$, $p = .6$; $t_{4yearolds}(20) = -4.5$, $p = .0002$ and $t_{5yearolds}(22) = -0.95$, $p = .4$, two-tailed, reflecting a bias on some children's part to simply choose the array of dots with the larger set size. Fig. 3c shows the proportion of children getting each test item correct in the Dots and Digits condition. The advantage of children's performance in mapping number names to digits over that of mapping those same number names to dot arrays holds for all multi-digit numbers tested: with the exception of the single digit comparison (3 versus 7), reliably more children responded correctly on all individual items in the Digits condition than the Dots condition, Chi-squares $(1) > 3.8$, $ps < 0.05$.

Altogether the results indicate the following: Many preschool children performed quite well in matching number names to written digits for 2- and 3-digit numbers. The ability of mapping number names to digits grew incrementally with age during this period. For all multi-digit numbers, performance was much better at all ages for Digits than for Dots. Overall, children's ability to map number names to dot arrays lagged far behind their ability to map number names to the written versions of those names. This was so despite the choice of comparisons expected to be hard when represented as digits but relatively easy given the actual quantities.

From the perspective of symbol learning, the advantage of Digits over Dots makes sense as both spoken and written number symbols are organized by the same base-10 principles and have the same underlying relational structure. Although children may never have been asked to map a heard number name such as "three hundred and five" to either the written form of this number or to a dot array, both number names and written digits have corresponding relational structures. Clouds of dots, in contrast, do not have any structure that aligns with that of the symbols through which we represent those quantities. From the perspective of the triple-code system in which perceived quantities provide the representational foundation for the other codes, the results are unexpected. Indeed, when children below the age of six were asked to map heard names to dots arrays, they performed at or below chance, unable to even map a 2-digit number to the quantity it represents when the foil contained 9 times as many dots. Preschool children apparently do not have even a rough sense of the physical quantities represented by these number names (see also, Sullivan & Barner, 2011). Four-year-old children consistently performed reliably below chance, choosing the larger quantity: When asked to choose "thirty six," for example, they chose the array with 306 dots rather than the one with 36. We conjecture that what 4-year-olds knew was that "thirty six" named a big amount—but had no representation of the magnitude of that quantity—and so they chose the array with more dots.

3. Experiment 2

Although children's ability to map number names to the written forms was consistently better at all ages than their ability to map number names to quantities, it still could be the case that the two developing abilities are related. Accordingly, Experiment 2 replicated Experiment 1 using a within-subject design to determine whether the age-related advances in the Digits task and the Dots task are correlated.

3.1. Method

Participants. Fifty-four children (26 male) were recruited from the same population as in Experiment 1, but none of the children participated in both experiments. There were 15 3-year-olds (mean age 41.3 months, range 36–48 months), 13 4-year-olds (mean age 51.6 months, range 48–57 months), 13 5-year-olds (mean age 64.02 months, range 60–71 months), and 13 6-year-olds, (mean age 75.6 months, range 72–81 months).

Stimuli and Procedure. All aspects of the stimuli and procedure were identical to Experiment 1 except that each child was tested twice, on separate days (separated by at least one day but no more than 10 days) in the Digits and Dots conditions. Across the entire sample, half the children were tested in the Digits task first and half in the Dots task first and within each age group, at least 6 children at each age level were tested with Digits first or with Dots first. The order of test trials was randomly determined for each child.

3.2. Results and discussion

Fig. 4 shows the overall pattern of performance for the four age groups and for the individual items. Regression analyses showed that performance in the Digits condition was strongly related to continuous age, $R^2 = 0.26$, $F(1,52) = 18.5$, $p < .001$, the same result found in Experiment 1. However, in this smaller sample study, performance in the Dots condition was not reliably related to age, $R^2 = 0.02$, $F(1, 52) = 1.02$, $p = .32$, whereas a weak relation was observed in Experiment 1. This weak correlation is likely due to the nonlinear relation between age and children's ability to map number names to dot arrays as evidence in Fig. 4b (and in Experiment 1 Fig. 3b); although 3-, 4- and 5-year-olds all performed at chance ($ts > -1.4$, $ps > 0.1$), the 3-year-olds on average had a higher number of correct trials ($M = 5.44$) than the 4-year-olds ($M = 4.42$) and the 5-year-olds ($M = 4$). Only the 6-year-olds perform above change ($t(12) = 3.3$, $p = .006$). A 4 (Age group) by 2 (Order of tasks) by 2 (Stimulus condition) analysis of variance for a mixed design yielded a main effect of Age group, $F(3, 46) = 6.92$, $p = .004$, partial $\eta^2 = 0.21$ and a main effect of Stimulus condition, $F(1, 46) = 33.92$, $p < .001$, partial $\eta^2 = 0.23$. The interaction between Age group and Condition approached conventional significance levels as the performance in the Digits task increased with

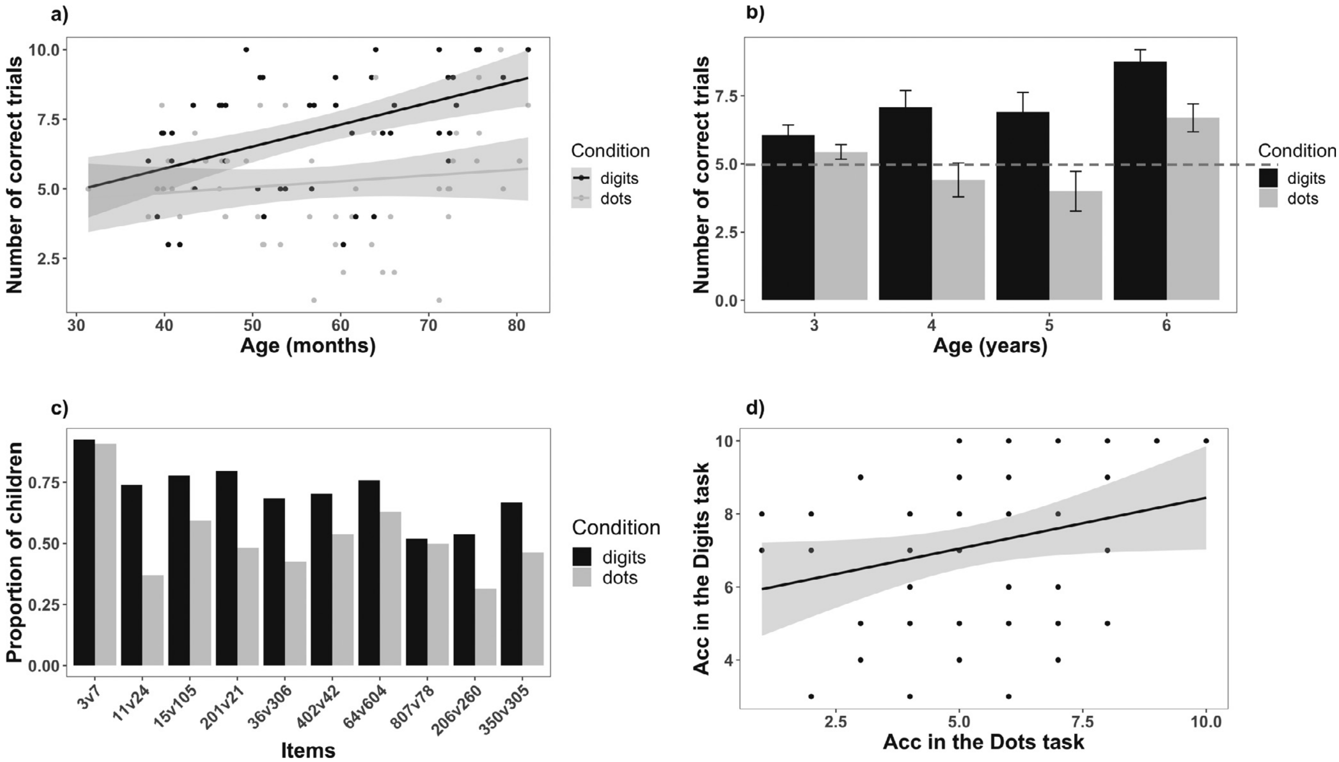


Fig. 4. Results from Experiment 2: (a) The number of trials with the correct answer for the Digits and Dots tasks as a function of age. (b) The average number of correct trials for the Digits and Dots task in each age group. The dotted line indicates chance performance. (c) The proportion of children who answered correctly for each of the test items in the Digits and Dots tasks. (d) The correlation between the number of correct trials in the Dots task and that in the Digits task.

age group more than did the performance in the Dots task, $F(3, 46) = 2.85$, $p = .048$, partial $\eta^2 = 0.06$. For the Digits task, all age groups performed above chance level, $ts > 2.7$, $ps < 0.02$. In contrast, for the Dots task, as described above, only the 6-year-olds performed above chance level.

The new question for this smaller sample within-subject replication of Experiment 1 was the relation between performance in the two tasks. Although knowledge of the mapping of number names to multi-digit numbers grows more steadily and rapidly during this developmental time frame than does knowledge of how these same number names map to dot arrays, it may still be the case that children who are better in one task are better in the other. As shown in Fig. 4d, this appears to be weakly the case, $r(52) = 0.27$, $p = .05$. However, if continuous age was entered first in a stepwise regression predicting performance in the Digits condition ($F(1, 52) = 18.467$, $p < .001$), performance in the Dots condition was excluded from the final model ($t = 1.749$, $p > .05$).

In sum, the overall pattern of children's performances in mapping spoken number names to multi-digit numbers and to dots arrays indicates: (1) During the preschool years, children's understanding of how spoken number names map to written multi-digit numbers increases systematically. (2) During this same period, children perform much more poorly and show much less systematic growth in their ability to map the same number names to dot arrays. (3) Children's performances in these two mapping tasks are not related beyond what can be explained by age alone. Overall, the results of Experiments 1 and 2 indicate that preschool children are building knowledge about how the two symbolic codes—names and written multi-digit numbers—map to each other but show less and later knowledge about how one symbolic code, multi-digit number names, map to perceived quantities. In brief, knowledge about how the two symbolic codes map to each other appears earlier and independent of children's ability to map these same number names to physical quantities.

4. Experiment 3

Preschoolers' ability to map number names to written forms in Experiments 1 and 2 implies at least an implicit knowledge about the relational principles behind the written notational system and names for multi-digit numbers, but does not necessarily signal any understanding of the meanings of those symbols. Children might only know how the two codes relate; and knowing what these symbols mean could depend on linking those symbolic codes to the specific quantities that they represent. This is a critical issue with respect to triple-code system (Dehaene, 1992; Dehaene & Cohen, 1995) as the perceived quantities are proposed to provide the semantic basis for the two symbolic codes of numbers. In Experiment 3, children were asked to make relative magnitude judgments, given digits or dots, in a between-subjects design. Given the large literature on children's competence in making relative magnitude judgments for perceived quantities (Halberda & Feigenson, 2008; Libertus et al., 2011), we expected children to perform well in the Dots condition. Can they also make these relative magnitude judgments in the Digits condition?

4.1. Method

Participants. The participants were 129 (64 male) children recruited from the same population as in Experiment 1. Some children (36) had participated in Experiment 1 at least 9 months prior to Experiment 3. Separate analyses of these children's performances indicated no differences in the pattern of results. There were 32 3-year-olds (mean = 41.6 mo, range 36 to 47 mo), 36 4-year-olds (mean = 53.6 mo, range 50 to 59 mo), 31 5-year-olds (mean = 64.1 mo, range 60 to 71 mo), and 31 6-year-olds (mean = 76.7 mo range 74 to 82 mo). Children at each age level were randomly assigned to either the Digits (n = 63) or Dots condition (N = 66) with roughly equal numbers of children within each age group assigned to each condition.

Stimuli and Procedure. The 10 trials were identical to those in

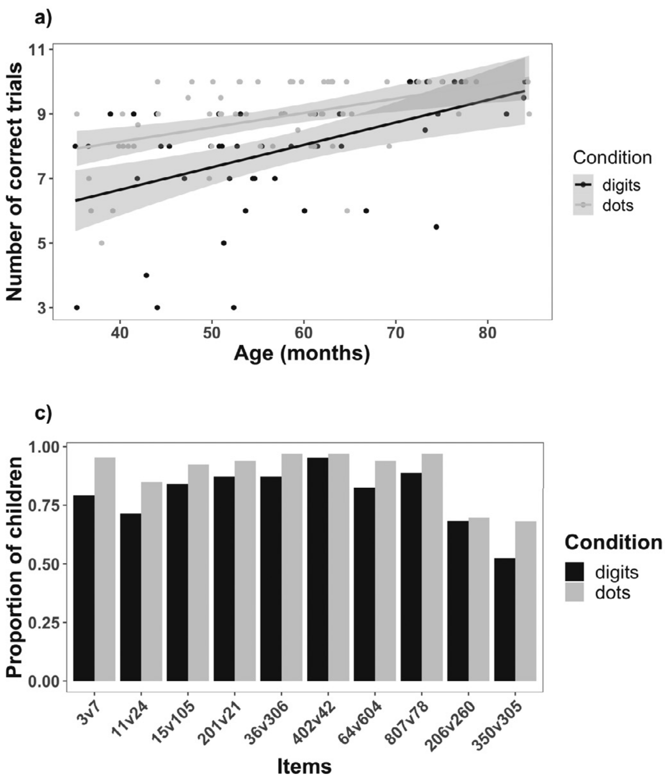


Fig. 5. Results from Experiment 3: (a) The number of trials with the correct answer for the Digits and Dots tasks as a function of age. (b) The average number of correct trials for the Digits and Dots task in each age group. The dotted line indicates chance performance. (c) The proportion of children who answered correctly for each of the test items in the Digits and Dots task.

Experiments 1 and 2 in both the Digits and Dots condition; the only difference was the question asked. The experimenter said "Look at these. Look at this one. Now, look at this one. Look at them both before you make your choice. Which one is more?" There were no warm-up trials. The procedure in both conditions began with the presentation of the two choice stimuli for the first trial. The order of the 10 trials was randomly determined for each child.

4.2. Results and discussion

Children performed quite well in both conditions, albeit performance was better in the Dots than Digits condition (89% vs 79%). As shown in Fig. 5a), correct responses were reliably related to continuous age for both the Dots condition, $R^2 = 0.19$, $F(1,64) = 14.5$, $p < .001$, and Digits condition, $R^2 = 0.19$, $F(1, 61) = 14.1$, $p < .001$. A 4 (Age group) by 2 (Stimulus condition) analysis of variance yielded only the two main effects of Age group, $F(3, 121) = 4.73$, $p < .01$, partial $\eta^2 = 0.12$ and Stimulus condition, $F(1, 121) = 13.89$, $p < .001$, partial $\eta^2 = 0.1$, with older children performing better than younger children and with performance in the Dots condition superior to performance in the Digits condition (see Fig. 5b). As shown in Fig. 5c, children performed well (> 75%) on all items except the comparisons of the two 3-digit-numbers (which had ratio differences near 1). Overall, quite young children performed very strongly in the Dots condition, which is consistent with what is known about the early development of perceived quantities and the greater than 9-fold difference between the target and the foil on 6 of the 10 trials. Children's success in making magnitude judgments also informs their failure (for children below age 6 years) to map number names to these same quantities. Although Experiment 3 shows that all children were well able to discriminate quantities such as 21 from 201, Experiments 1 and 2 showed that children of the same age group do not know that "twenty-one" is the name for an array of 21 dots, a failure that apparently does not reflect difficulties in discriminating the quantities.

The new finding is that children also responded well in the Digits condition. Apparently, their emerging knowledge of the written multi-digit numbers includes at least partial knowledge of how these symbols are ordered by relative magnitude, a beginning sense of their meaning. However, given the items used in Experiment 3 (borrowed from Experiments 1 and 2), children's success may be due to rather unsophisticated understanding of how the written forms relate to magnitudes. In the Digits condition, children could respond correctly on 6 of the 10 items simply by knowing that more digits (3 digits versus 2 digits) indicate bigger amounts. However, as indicated in Fig. 3c, the number of children performing above chance on either or both of the two items with the target and the foil composed of 3 digits (206 vs. 260, 350 vs. 305) is greater than expected by chance (chi-square (1) = 4.5, $p = .03$). Although the 206 vs. 260 comparison ($M = 68.2\%$, two-sided binominal test: $p = .005$) appeared to be easier than the 350 vs. 305 comparison ($M = 52.3\%$, two-sided binominal test: $p = .8$), the overall result suggests that preschool children may be developing more sophisticated knowledge than simply counting the numbers of digits, a hypothesis that we further test in the next experiment.

5. Experiment 4

In Experiment 4, we used more challenging comparisons to probe children's knowledge about the relative magnitudes of multi-digit numbers, using a within-subject design to also determine whether children's magnitude comparisons of written multi-digit numbers are independent of or related to their abilities to make relative magnitude judgments of the same quantities in dot arrays.

5.1. Method

Participants. The participants were 62 (31 male) children recruited from the same population as the previous experiments and none had participated in any of the prior experiments. There were 13 3-year-olds (mean = 40.7 mo, range 36 to 47 mo), 19 4-year-olds (mean = 53.7 mo, range 48 to 58 mo), 16 5-year-olds (mean = 65.1 mo, range 60 to 70 mo), and 14 6-year-olds (mean = 75.8 mo range 72 to 81 mo). Children in each age group were randomly assigned to either the Digits or Dots condition as the first tested condition. Children were tested in the two conditions on separate days at their daycares or nursery schools with at least one day but no more than 10 days separating the two testing sessions.

Stimuli. The 10 comparison trials were constructed as in Experiment 3 and consisted of the following three kinds: (1) 2-digit number comparisons—11 v 24, 16 v 23, 30 v 60; (2) 2- v 3-digit number comparisons—15 v 105, 21 v 201, and (3) 3-digit number comparisons—220 v 223, 321 v 323, 525 v 585, 305 v 350, 206 v 260. The 3-digit comparisons differ in one-place—the tens or ones—or are transpositions of the ten's and one's place. These 3-digit number comparisons provide a strong test of children's ability to make magnitude judgments given the written forms.

Procedure. There were no warm-up trials. The procedure in both conditions began with the presentation of the two choice stimuli for the first trial. The experimenter said "Look at these. Look at this one. Now, look at this one. Look at them both before you make your choice. Which one is more?" The order of the 10 trials in each condition was randomly determined for each child.

5.2. Results and discussion

Fig. 6a shows individual performance in the Dots and Digits conditions as a function of continuous age. Performance in the Dots condition was only weakly related to continuous age, $R^2 = 0.09$, F (1, 60) = 5.57, p = .02, as most children performed above chance, t (61) = 15, p < .0001. Performance in the Digits condition, in contrast, was strongly related to age, $R^2 = 0.41$, F (1, 60) = 41.69, p < .0001. In this harder test of the relative magnitude meaning of written digits, younger preschoolers performed quite poorly but older preschoolers performed quite competently. An analysis of variance for 4 (Age group) by 2 (Stimulus condition) by 2 (Order of Tasks) mixed repeated measure design revealed a main effect of Age group, F(3, 54) = 13.46, p < .001, partial $\eta^2 = 0.31$, and a main effect of Stimulus condition, F (1.54) = 4.58, p < .05, partial $\eta^2 = 0.22$. The interaction between Age and Condition was also significant, F(3,54) = 3.01, p < .05, partial $\eta^2 = 0.09$; as shown in Fig. 6b, the difference between performance in the Dots and Digits condition declined with age. No other main effects or interactions approached significance. Fig. 6b also shows that children's overall performances in both conditions were quite strong after 5 years of age. In particular, for the Digits condition, all age groups except for the 3-year-olds (t (12) = 1.4, p = .2) performed significantly higher than chance (ts > 3.3, ps < 0.004). As shown in Fig. 6c, this includes many of the 3-digit number comparisons; these require an understanding of the magnitude implications of places in the representational system. However, performance in the two tasks were unrelated, r (60) = 0.03, p = .8. As is apparent in Fig. 6d, children with the same level of performance in the Dots condition varied in their performance in the Digits conditions from quite poor to perfect. These results indicate that preschool children's emerging knowledge about the symbol system ultimately goes beyond mapping names to written numbers (certainly by 5 years of age) to include an initial sense of the meaning, the relative magnitudes indicated by the symbols. This emerging knowledge appears unrelated to and is not predicted by their ability to make relative magnitude comparisons of the physical quantities.

The results of Experiments 1 to 4 indicate that preschool children are developing knowledge of how written multi-digit numbers represent large quantities and that these developments appear unrelated to their abilities to directly compare the perceptual quantities or to map

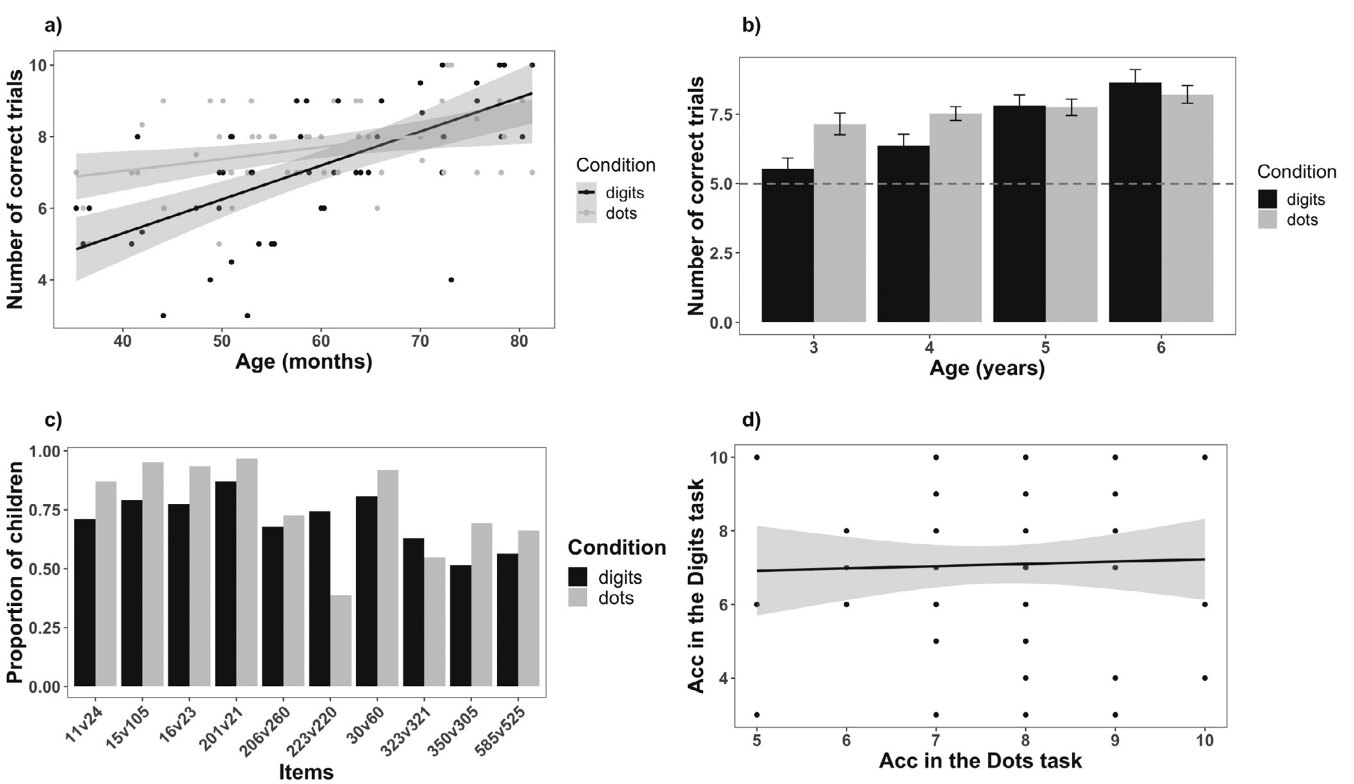


Fig. 6. Results from Experiment 4: (a) The number of trials with the correct answer for the Digits and Dots tasks as a function of age. (b) The correlation between the number of correct trials in the Dots task and that in the Digits task. The dotted line indicates chance performance. (c) The average number of correct trials for the Digits and Dots task in each age group. (d) The proportion of children who answered correctly for each of the test items in the Digits and Dots tasks.

number names to those quantities. In all four experiments, performance in the Digits task was strongly related to continuous age whereas performance in the Dots task was not or weakly related to age. This fact makes sense if these are two independent paths to understanding large numbers. Knowledge about multi-digit notation must emerge from exposure to heard number names and written digits; it is knowledge about a cultural artifact and thus experience in the world is a driver of this knowledge. Perceiving and comparing large sets of objects is, in contrast, a core perceptual ability (Dehaene, 2011; Halberda et al., 2008) that could be more strongly influenced by intrinsic individual differences than by experience. The main conclusion from Experiments 1 – 4 is this: There is a route to understanding large numbers that begins early and concerns knowledge about the symbol system, and is not strongly related to the ability to perceive and judge those same quantities as sets of things.

6. Experiment 5

Experiment 5 was designed to deepen our understanding of children's beginning knowledge of the symbol system—names and written forms—in two ways. First, we included even more challenging comparisons. Second, we examined within-subjects the relation between children's mapping of number names to written digits and their ability to make relative magnitude judgements given just the written form, making the final point that emerging knowledge of the two symbolic codes—names, written forms, and their relative magnitudes—are tightly connected, robust when tested with a variety of numbers, and strengthening steadily over the preschool period.

6.1. Method

Participants. The participants were 54 preschoolers (28 male) recruited from the same population as Experiments 1 to 4:14 3-year-olds (mean 41.5, range 36 – 47 months), 13 4-year-olds (mean 51.9, range 48 to 58 months), 13 5-year-olds (mean 64.1, range 60 to 71 months), and 14 6-year-olds (mean 75.6, range 72 to 75 months). Children were tested in the laboratory or at preschools. None had participated in the previous experiments.

Stimuli. The stimuli were written numbers**.** The 15 comparisons for the "which-More" task were: 3 v 7, 6 v 8, 11 v 19, 14 v 41, 16 v 62, 26 v 73, 30 v 60, 72 v 27, 100 v 10, 101 v 99, 123 v 321, 220 v 223, 525 v 585, 670 v 270 and 4620 v 4520. The "which-N" trials included some of the comparisons used in the previous experiments but also added several new items to add converging evidence on preschool children's competence in mapping number names to written digits. The 16 items were: 2 v 8, 11 v 24, 12 v 22, 15 v 5, 21 v 201, 36 v 306, 42 v 402, 64 v 604, 85 v 850, 105 v 125, 305 v 350, 206 v 260, 670 v 67, 100 v 1000, 807 v 78, 1002 v 1020. (There is one fewer item in the "which-More" task than "which-N" task because of an error in the construction of the "which-More" test set).

Procedure. The design and procedures for the "which-N" and "which-More" task were identical to those used in the previous experiments. Children were tested on the same day with a break between tasks. Half were tested on "which-N" first and half were tested on "which-More" first.

6.2. Results and discussion

In Fig. 7a, the grey dots show scatterplots of children's performances on the which-N task as a function of age and the black dots show performance on the which-More task as a function of age. Performance on both the which-N and which-More tasks were strongly related to age (R^2 (52) = 0.30, $p < 0.001$; R^2 (52) = 0.31, $p < 0.001$). Performance on these two tasks were also strongly correlated to each other (r (52) = 0.56, $p < 0.001$), Fig. 7e.

Table 1 shows the results of two multiple regression models: (a) Age and performance in the which-N task predicting performance in the which-More task and (b) Age and performance in the which-More task predicting performance in the which-N task. The which-N task uniquely explained 31% of the variance in the which-More task, and the which-More task uniquely explained 30% of the variance in the which-N task. A 4 (Age group) by 2 (Task) by 2 (Task order) ANOVA revealed a significant main effect of age, F (3, 46) = 11.66, $p < .0001$. partial $\eta^2 = 0.33$. There was no main effect of task, F (1, 46) = 2.76, p = .1 or task order F (1, 46) = 0.01, p = .9. Although these analyses cannot tell us precisely how performance in the two tasks develop, they suggest that they are developing as an early unified system during the preschool years: children's knowledge about the names of and relative magnitudes represented by written multi-digit numbers grow steadily and jointly in 3- to 6-year-olds as shown in Fig. 7b.

7. Aggregated data analysis

To more fully understand what children know about multi-digit numbers and the quantities they represent, we combined data from all five experiments and characterized each test item according to four mutually exclusive categories. As shown in Table 2, the four categories are as follows, quantities written as: Single-digit numbers (S, e.g., 3 v 7); Multi-digit numbers with different places (MD-DP, e.g., 42 v 402); Multi-digit numbers with the same number of places but no transposition (MD-SP-no-T, e.g., 525 v 585). Multi-digit numbers with the same number of places and transpositions (MD-SP-T, e.g., 305 v 350). To correctly answer items that involve transpositions, children need to know that number words are mapped onto written numbers following the left-to-right order and that the left-most-digits matter more for the magnitude of numbers.

Fig. 8a shows the accuracy of different types of items in the which-N task with digits. As can be seen, children of all age groups performed well (above 70%) with single digits. By age 4, children performed above chance (t (45) = 4.13, p = .0001) on multi-digits items that involve the same number of places but no transposition (e.g., 525 v 585). To succeed on these items, children must know, at the minimum, how the variants of number names for different places map to the written digits. For example, if they had heard "five-hundred twenty-five", they could infer that the correct number must have the digit "2" rather than "8" in the written form because "twenty" also names "2.". Around the same age, children show increasing success in mapping names to digits that differ in the number of places (t (45) = 2.79, p = .007), although performance is slightly lower than with single digits (t (75) = 2, p = .048). Success on these items shows a beginning understanding of places, that "3-hundred" and "thirty" both refer to a "3" but signal different spatial locations of that 3 in the written form. Multi-digit numbers with the same number of places and transpositions require that children know that the temporal order of number words are mapped onto a left-to-right spatial order of written numbers: it is not until age 5 that children performed above chance level (t (45) = 2.89, p = .005) on these items. By age 6, children performed well (> 75%) on all item types.

Fig. 8b shows the accuracy of different types of items in the which-More task with digits. Here, the only category on which 3-year-olds performed above chance (t (48) = 7.81, $p < .001$) was multi-digit numbers with different places (e.g., 42 v 402), suggesting that children's first approach to understanding the magnitudes of written forms may be based on the number of components or the amount of written stuff, an approach seen in and perhaps borrowed from children's early ideas about words (Bialystok, 1992). By age 4, children also performed above chance on single digit number comparisons (t (29) = 4.01, $p < .001$). The difference in the which-N and which-More task for single digit numbers suggest that children learn to map number names to written digits before they understand the quantities that those digits represent (see also, Hurst, Anderson, & Cordes, 2016; Sullivan & Barner, 2014). By age 5 years, children performed above chance on multi-digit

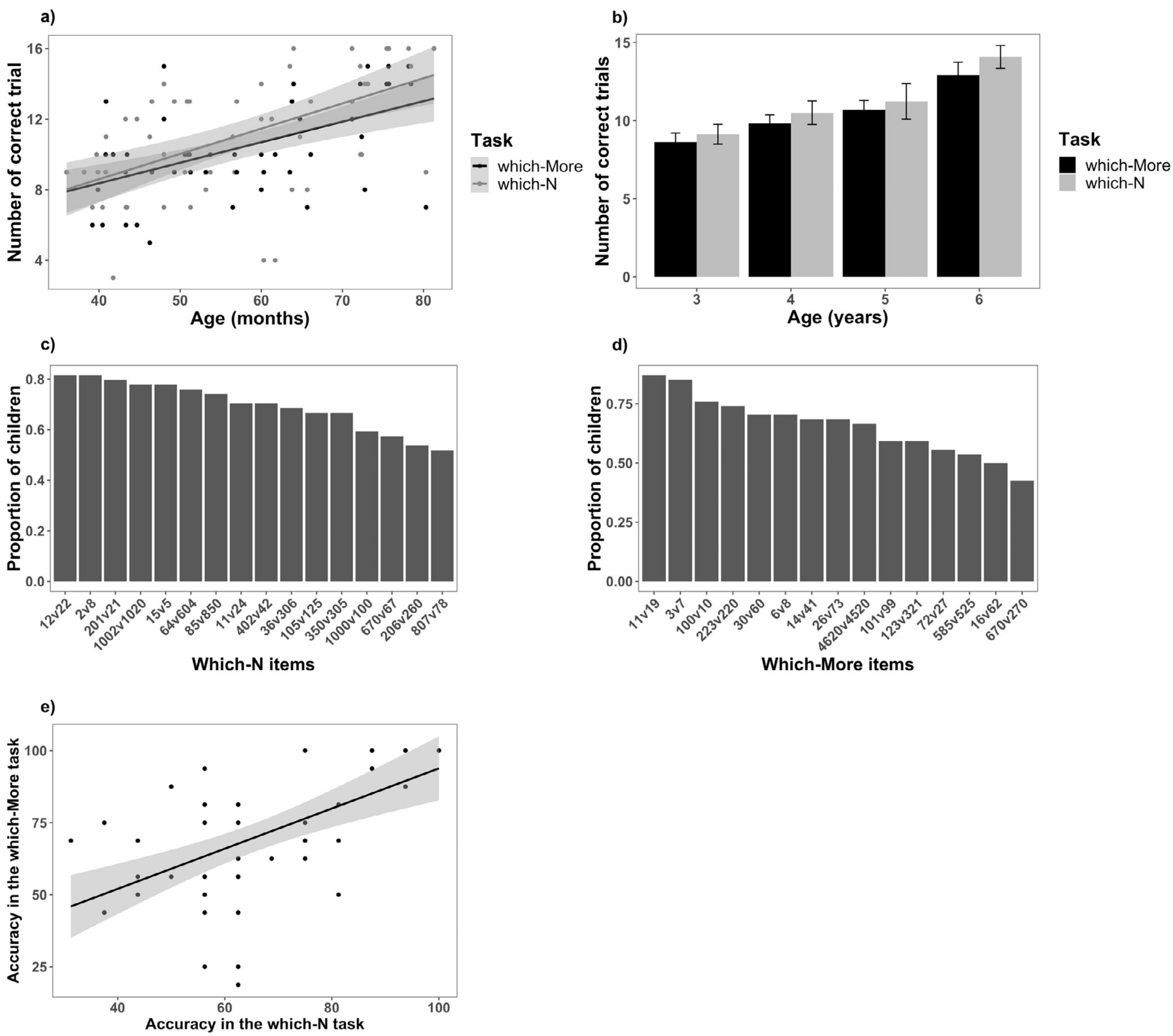


Fig. 7. Results from Experiment 5: (a) The number of trials with the correct answer for the which-N and which-More tasks as a function of age. (b) The average number of correct trials for the which-N and which-More task in each age group. Chance performances were 7.5 for the which-More task and 8 for the which-N task. (c) shows the proportion of children who answered correctly for each of the test items in the which-N task. (d) The proportion of children who answered correctly for each of the test items in the which-M task. (e) The correlation between the number of correct trials in the which-N task and that in the which-More task.

numbers with same places but no transposition ($t(43) = 9.74$, $p < .001$); and it was not until age 6 that they did so with multi-digit numbers that involved transposition ($t(42) = 9.91$, $p < .001$). Overall the results in the Digits tasks suggest that links between number names and written forms of multi-digit numbers are formed early and grow incrementally with knowledge about the relative magnitudes represented by the symbols developing closely behind. A more direct test of this last point is clearly required before strong conclusions are made.

Performances in the Dots tasks show very different developmental patterns than those in the Digits tasks. For the which-N task, as shown in Fig. 8c, children younger than 6 years of age only make accurate mappings for single digit numbers; performances on all the other categories were at or below chance for 3-, 4- and 5-year-olds. For the which-More tasks, in contrast, all children, even 3- year-olds, perform above chance across all item types ($ts > 3.3$, $ps < 0.002$) as shown in Fig. 8d. Prior to 6 years of age, preschool children can make competent judgments about physical quantities represented by multi-digit numbers, but they apparently have little knowledge about how these specific quantities link to the numbers that represent them.

The four categories of trial types are based on the properties of the notational system and thus have little to do with the relevant properties that underlie children's judgments of actual quantities, which are based on set size (Halberda & Feigenson, 2008; Lyons, Nuerk, & Ansari, 2015).). Accordingly, we also analyzed performances in the Dots and Digits tasks according to the ratio between the set sizes of the to-be-compared items, focusing solely on the multi-digit numbers because previous research has suggested that physical quantity representation might be significantly different between small and large arrays (Sullivan & Barner, 2014), and because children's knowledge of multi-digit is the current focus. Consider the pattern of performance for the dot arrays first. In the which-More task—a form of magnitude comparison which has been widely studied with dot arrays—we find what has been reported many times (Dehaene, 2007; Halberda & Feigenson, 2008): For all age groups, overall performance is strong but decreased as the ratio between the two numbers increased (Fig. 9a). The which-N task, which requires children to link a symbolic code to a perceived

Table 1
Two multiple regression models: (a) Age and Accuracy at the which-N task predicting Accuracy at the which-More task; (b) Age and Accuracy at the which-More task predicting Accuracy at the which-N task.

	B	*SE B*	*β*
(a)			
Model 1:			
Dependent variable:			
Accuracy at the More task			
Predictor variables:			
Constant	17.962	8.511	
Age	0.466	0.169	0.359**
Accuracy at the N task	0.292	0.105	0.362**
(b)			
Model 2:			
Dependent variable:			
Accuracy at the N task			
Predictor variables:			
Constant	7.476	11.009	
Age	0.564	0.211	0.35**
Accuracy at the More task	0.454	0.162	0.365**

quantity, shows a very different pattern. As shown in Fig. 9b, all age groups except 6-year-olds performed poorly at all ratios. The oldest children were much more successful, performing well above chance with most of the ratios and only showing an effect of ratio when it approached 1 (and the limit of perceptual discriminability). Although earlier developing symbol knowledge appears unrelated to the perception of the physical quantities, 6-year-olds appear to be forming an integrated triple-code system that pulls together developmentally prior and separate knowledge about big numbers.

Fig. 9c and d shows the performances of children in the Digits task as a function of the ratio difference of the represented quantities. Just as the categories of symbolic forms tell us little about children's judgements of physical quantities, ratio differences appear to have little relevance to children judgments of the written forms, both in mapping names to those written forms and in making magnitude judgements. These findings provide further evidence of the early independence of learning about multi-digit numbers and the perception and discrimination of the physical quantities represented by those numbers.

In summary, the analyses of the aggregated results indicate the following: Children's understanding of the two symbol system codes—names and written forms—show incremental development characterized not by the magnitude of physical quantities represented by the multi-digit numbers but by increasing knowledge about the codes themselves. Children's perceptual discrimination of the physical quantities represented by multi-digit numbers is limited only by the ratio difference between the judged magnitudes and appears quite independent from the structure of the notational system. For multi-digit numbers (but perhaps not for single-digit numbers), the perceptual code in the triple-code system appears unlinked to the symbolic codes until late in the preschool years—emerging in 6-year-olds—and well after preschool children can map number names to written forms and make magnitude judgements given only the written forms.

8. General discussion

Considerable evidence indicates that understanding the base-10 notational system is central to success in elementary school mathematics (Anderson, 2013; Mix et al., 2019; Wai, Chan, Au, & Tang, 2014; Zuber et al., 2009). Considerable research also indicates that place value concepts are difficult to master for many children late into elementary school (Ross, 1986; Fuson & Briars, 1990; Fuson, 1990; Gervasoni et al., 2011; Ross & Sunflower, 1995). Accordingly, much previous work focused on older children (2nd – 5th grade) and their misunderstandings of place value (Cobb, 1988; Kamii, 1988; Ross, 1986; Ross & Sunflower, 1995 with little research on what preschoolers might know about multi-digit numbers. The first contribution of the present findings is the documentation of early knowledge about the names, written symbols and relative magnitudes of multi-digit numbers, findings that have direct implications for understanding why mastering place value is hard for some (but not all) children. The second contribution is that the experiments show that this early knowledge about names and written forms is not built on the mapping of those symbols to the quantities they denote, a finding with implications for the triple-code system and its development. The following discussion first considers the implications with respect to the problem of children's mastery of place value concepts and then the implications with respect to the hypothesized triple-code system.

8.1. Preschoolers and place value

Many of the children in the five experiments clearly knew how number names map to written forms and could make magnitude judgments given written symbols. Clearly, they had—in some manner—induced the general principles through which they could make judgments of specific 3- or 4-digit numbers even though they were unlikely to have had but a few (if any) encounters with these specific multi-digit numbers. By the time children were five years of age, most knew several structural regularities: that the first-mentioned number name corresponds to the left-most written digit, the magnitude indicated by each place decreases from left to right, "five hundred", "fifty", and "five" are all represented by "5" and that variations in the names associated with the same digit signal different places and different magnitudes. It seems likely these principles are initially derived based on the surface properties of the symbols, including thinking that numbers with more digits represent greater magnitudes, and noticing that the 5 in 500 looks the same as the 5 in 50. Such beginning ideas are *not* the complete knowledge necessary for success in multi-digit calculation which requires an explicit understanding of the multiplicative hierarchy underlying base-10 notation—that the hundreds place counts sets of tens, the tens place counts sets of ones, and that 100 is 10 tens, and 10 is 10 ones, and so forth. Given the difficulty of these explicit concepts for elementary school children, it is highly unlikely that any of the children in the present experiments have an explicit understanding of base-10 principles. But the abilities that preschoolers showed in making correct judgments about spoken and written numbers is a start and one not apparently dependent on linking symbols to the quantities they represent.

Table 2
Items and their corresponding categories.

Category	Items
Single-digit numbers (S)	3 v 7, 6 v 8, 2 v 8
Multi-digit numbers with different places (MD-DP)	15 v 5, 15 v 105, 21 v 201, 36 v 306, 42 v 402, 64 v 604, 78 v 807, 100 v 10, 101 v 99, 85 v 850, 670 v 67, 100 v 1000
Multi-digit numbers with same number of places but no transposition (MD-SP-no-T)	11 v 24, 16 v 23, 30 v 60, 220 v 223, 321 v 323, 525 v 585, 11 v 19, 16 v 62, 26 v 73, 670 v 270, 1620 v 1520, 12 v 22, 105 v 125
Multi-digit numbers with same number of places and transpositions (MD-SP-T)	206 v 260, 305 v 350, 14 v 41, 72 v 27, 123 v 321, 1002 v 1020

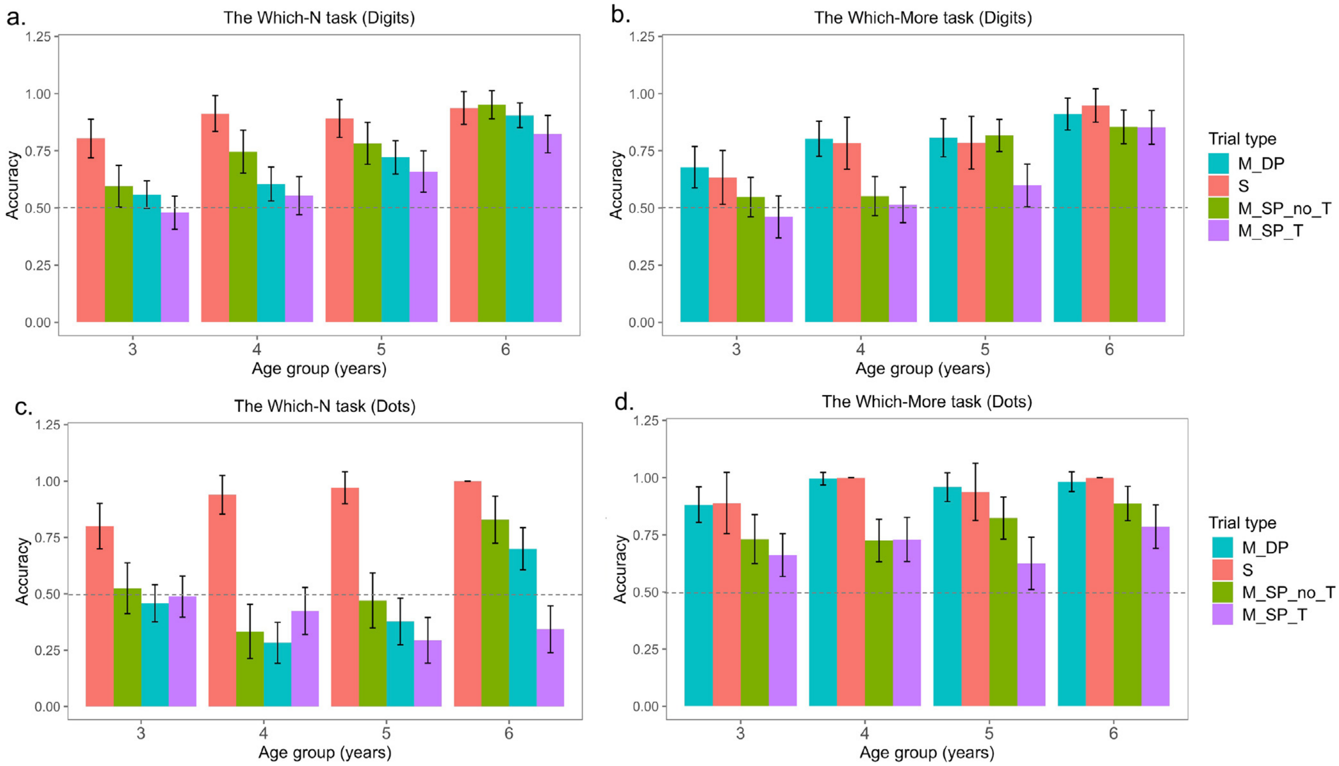


Fig. 8. Accuracy by item type, task and age group: (a) accuracy in the which-N task with digits; (b) accuracy in the which-More task with digits; (c) accuracy in the which-N task with dots; (d) accuracy in the which-More task with dots. The gray dotted line represents the chance level.

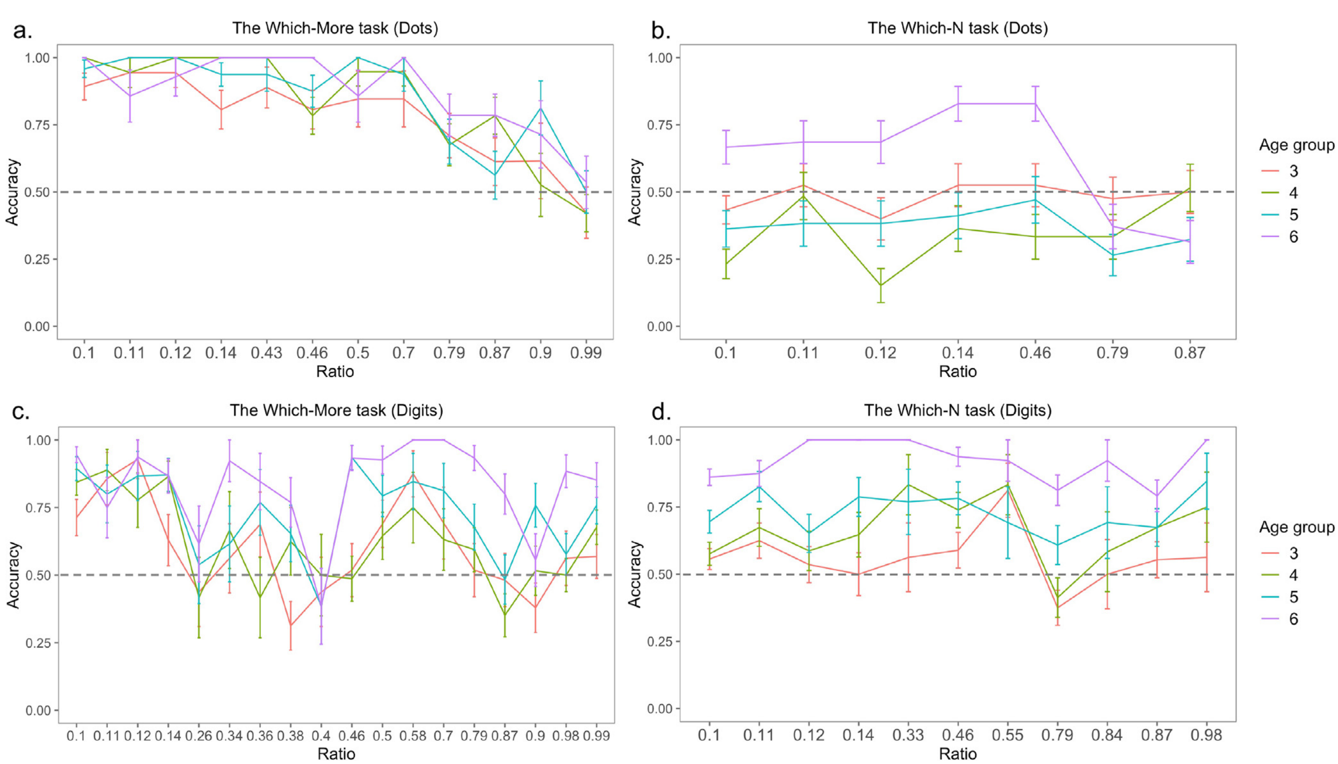


Fig. 9. Performance in the which-More and which-N task with dots and digits as a function of the ratio between the two choices: (a) accuracy in the which-More task with dots; (b) accuracy in the which-N task with dots; (c) accuracy in the which-More task with digits; (d) accuracy in the which-N task with digits. The gray dotted line represents the chance level.

It seems highly likely that preschool children's early (albeit incomplete) knowledge of the symbol system plays a role in how well they learn from explicit instruction about base-10 notation. In the classroom, formal instruction typically includes talk about multi-digit numbers in highly cluttered contexts that include multiple written multi-digit numbers along with physical models of sticks or blocks being grouped and ungrouped. Prior knowledge about multi-digit number names, how they map to written forms, and their relative magnitude may help guide visual attention to the right referents at the right time (Huettig, Rommers, & Meyer, 2011; Tanenhaus, Spivey-Knowlton, Eberhard, & Sedivy, 1995). Facility with spoken number names and written representations may also support the forming of robust and correct memories of classroom instruction and in doing so may prevent the formation of wrong ideas that characterize some children even as late as sixth grade (Gervasoni et al., 2011; Ross & Sunflower, 1995). Thus, early learning about the surface properties of the naming and notational system may support later learning about place value in the same way that building perceptual fluency in recognizing the structure of an equation or how a function relates to a graph has been shown to advance learning in higher mathematics (Goldstone, 1998; Kahnt, Grueschow, Speck, & Haynes, 2011; Kellman, 2002; Kellman, Massey, & Son, 2010; Landy & Goldstone, 2007).

Preschool knowledge about the symbol system may also play a more direct role in the formation of the core principles of base-10 notation. We know from computational models (Grossberg & Repin, 2003; Rule et al., 2015) that the mappings of number names to written forms instantiate deeper latent knowledge about base-10 principles. This implicit knowledge could be prerequisite to an explicit understanding of base-10 notation. This hypothesis is motivated by analogy as to how humans learn explicit concepts of syntax. Speakers of English, in their everyday use of language, show an implicit understanding of syntactic categories such as noun phrase and verb phrase. This implicit knowledge can be made explicit through formal instruction in linguistics. Critically, prior intuitive knowledge holds the meaning to which explicit categories such as "noun phrase" or "verb phrase" refer because those categories do *not* refer to perceptible entities but rather to the relational structure of variables within the syntactic system. Mathematical knowledge about base-10 principles is arguably similar; the meaning of places does not lie in any specific quantity nor in bundled sticks arranged in sets of 10 but in *the relational structure* among places that is base-10 notation. If this idea is right, then successful instruction in making those relational concepts explicit may depend on the hidden latent knowledge about place value that emerges from the mappings of names to written forms.

Both of these hypotheses—perceptual fluency and latent knowledge—along with the evidence of preschool children's early learning about the symbols suggest that solutions to school-age children's difficulties in mastering place value may be found by understanding what children know before school. The findings across the five experiments show not only incremental growth in knowledge about multi-digit numbers during the preschool years but also marked individual differences. Some children as young as three years of age already knew a lot about how number names map to written forms and the relative magnitudes they signify. Some children as old as 5 and 6 years, however, performed quite poorly on the which-N and which-More tests. Understanding how these individual differences play out in the context of formal instruction will require that we go beyond the which-N and which-More to direct assessments of perceptual fluency, robustness of memory, and the possible engagement of latent knowledge about base-10 principles in the context of explicit instruction.

If early knowledge about the multi-digit numbers supports later mastery of place value, then we also need to determine the kinds of experiences that give rise to this early knowledge. Past research provides two relevant pieces of information. First, we know that preschool children whose parents or teachers talk about numbers—in a variety of contexts—have greater success in formal learning about numbers (Levine et al., 2010). Second, we know that the amount of talk about multi-digit numbers in everyday conversations with children is extremely sparse (Dehaene, 1992; Dehaene & Mehler, 1992; Levine et al., 2010). Thus, children with more number talk in their environments are likely to acquire knowledge about the symbol system earlier than those without such talk, but even the children in the richest number environments may not be hearing individual multi-digit names or seeing their written forms with great frequency. Thus, it would seem that preschool children must be learning the principles behind number names and written forms through encounters with a sparse sampling of names and numbers that occur with relatively low frequency compared to other forms of talk. How might this work?

Number names and written digits comprise two parallel relational structures of the kind studied within the framework of Structure Mapping (Gentner, 1983, 2010; Gentner & Colhoun, 2010. Given varying but relationally alignable surface forms—for example, models of the solar system and atoms or relational series such as big-little-big—learners can discover the relational structure and apply that structure to new instances (Gick & Holyoak, 1983; Goldwater, Bainbridge, & Murphy, 2016; Loewenstein, Thompson, & Gentner, 1999). Moreover, research with children as well as adults indicates that relational structures may be learned from exposure to two alignable series, without explicit teaching or feedback (Christie & Gentner, 2010; Fisher, 1996; Gentner et al., 2016; Namy & Gentner, 2002). We conjecture that the alignable structure of number names and written forms may be key to the early knowledge about multi-digit numbers shown by the preschoolers in the present study. Relatively sparse encounters—from reading calendars, street addresses, price tags in the stores, learning the counting list, games such as Pokémon—may, because of the alignable structure, yield latent (and thus generalizable) knowledge about the underlying relational structure of places and their values. Although many researchers have pointed to the non-alignability of names and numbers across the teens (Fuson & Kwon, 1991, 1992; Geary, Bow-Thomas, Liu, & Siegler, 1996; Ho & Fuson, 1998; Bussi, 2011; Saxton & Towse, 1998), there apparently is sufficient evidence in many preschool children's experiences to induce some principles about how the names map to written forms and how written forms represent relative magnitudes (see also, Hurst et al., 2016; Sullivan & Barner, 2014). The experiences and mechanisms through which many but not all preschool children develop the abilities documented in the present studies is a critical target for future research.

8.2. The triple-code system

The perception and representation of discrete quantities has been conceptualized as the foundation of the triple-code system (Dehaene, 1992; Dehaene & Cohen, 1995) with the specific proposal that the symbols represent and gain their meaning through perceptual representations of physical quantities. Children entering school with robust understanding of the links among all three components of the tripe code for single-digit numbers—their written forms, names and specific quantities (Hurst et al., 2016). Multi-digit notation, however, represents much more than physical quantities; it represents a set of relations among all possible numbers. The present results suggest that learning about these big numbers does not begin with a mapping of specific numbers to the quantities they represent, but begins with learning about the relational patterns in the surface forms of names and written digits.

One potential limitation in the current study is that we asked to children to judge physical arrays of quantities that like those in the world do not control for the many dimensions (e.g., area, density, contour length) that co-vary in with number (Cantrell & Smith, 2013). Previous studies measuring the Weber's law in terms of the ratio difference between quantities require confidence in that the perceptual discrimination is based on discrete number alone and so attempted to control for the many other discriminated quantities (with variable results, Cantrell & Smith, 2013; Clayton, Gilmore, & Inglis, 2015; Smets, Sasanguie, Szücs, & Reynvoet, 2015). We did not include these controls because our research question was not about the acuity of children's number description but about whether knowledge about the spoken and written symbols depended on having them mapped to the physical quantities they represented. As a result, we may have *over-estimated* children's performance in the non-symbolic tasks. Given past research (see especially, Clayton et al., 2015; Smets et al., 2015), it is likely that adding a stricter control in the current study would result in even worse performance in the dots tasks. Although not tested here, this possibility would lend more support for the argument that the understanding of multi-digit symbolic numbers is not based on non-symbolic representation.

Young children's integrated triple-code knowledge of single digits likely plays an important role in the discovery of these relational patterns in multi-digit numbers. Children's knowledge of the quantities 1 through 10, through direct perception (Huang, Spelke, & Snedeker, 2010; Odic, Le Corre, & Halberda, 2015; Pinhas, Donohue, Woldorff, & Brannon, 2014; Wagner & Johnson, 2011) or counting (Carey, 2001; Gentner, 2010; Le Corre & Carey, 2007) is likely essential to understanding the relative magnitudes of multi-digit numbers, and since places *count* sets of multiples of tens, the determination of exact quantities for counts from none to 10—through counting (Gallistel & Gelman, 1992; Wynn, 1992) or by linking names to perceptual representations via processes such as subitizing, pattern recognition or object files—will be essential to an explicit understanding of base-10 principles. Children's abilities to determine the exact quantity of small set sizes have also been shown to be related to children's developing understanding of basic numeracy principles, including cardinal concepts of numbers as well as addition and subtraction concepts (Carey, 2001, 2010; Le Corre & Carey, 2007; Libertus et al., 2011; Mazzocco et al., 2011; Park & Brannon, 2014; Sullivan & Barner, 2014) and these must be generalized to very large numbers. But the present results also clearly show that children's early understanding of large numbers does not depend on linking the symbols to the exact—or approximate—quantities they signify. However, the present findings suggest that 6-year-olds are at least beginning to integrate the two systems. Given this integration, the open question is how symbol knowledge influences perceptual representations and how perceptual representations influence symbol knowledge. The extant evidence strongly suggests influences in both directions, although their nature is not well-understood (Landy, Charlesworth, & Ottmar, 2016; Lyons, Ansari, & Beilock, 2012; Mussolin et al., 2014; Park & Brannon, 2014; Sullivan & Barner, 2014; Thompson & Opfer, 2010). All of this suggest that the development of the triple-code system is likely to be far more complex—and nuanced—than that of providing a direct perceptual grounding for the symbols. Other evidence also suggests a complicated developmental trajectory in that training in solving math problems benefits judgments of physical quantities and training perceptual judgements of physical quantities benefits mathematics problem solving (Lyons, Bugden, Zheng, De Jesus, & Ansari, 2018; Park & Brannon, 2013).

Nonetheless, the importance of the triple-code system may be for very early entry into learning about numbers, and perceptual judgments of physical quantities and mathematics itself may depend on fundamentally different skill sets. Elementary school mathematics—arithmetic—is about determining exact quantities. But mathematics proper is not; it is instead about systems of relations among quantities. There are also some empirical indicators that symbolic skills may be the more critical factor in later achievements. Meta-analyses of the relation between older children's performances in symbolic and non-symbolic number tasks with later mathematics achievement (Chen & Li, 2014; Fazio, Bailey, Thompson, & Siegler, 2014; Schneider et al., 2017) indicate that symbolic knowledge is strongly related to later mathematics achievement (Göbel, Watson, Lervåg, & Hulme, 2014), but that non-symbolic knowledge is only weakly related. Further, in some analyses of these predictive relations to later mathematics achievement, performance in symbolic and non-symbolic tasks have been found to load on different factors, and there are limited or no correlation between symbolic and non-symbolic magnitude knowledge (Lyons et al., 2012; Sasanguie, Defever, Maertens, & Reynvoet, 2014), just as observed here. The new contribution of the present study is that it shows this same non-relation in preschool children who show early competence in learning about the relational structure in symbolic representations of numbers larger than 100. The findings bring us to new questions and emphasize the importance of understanding what preschool children know about the notational system, the learning environments that support this early knowledge, and its consequences for later mathematics learning.

References

Anderson, A. (2013). A practical guide to early numeracy instruction for general and special educators. *Journal of Education and Training, 1*(1 SRC-GoogleScholar FG-0), 11–18.

Barth, H., La Mont, K., Lipton, J., Dehaene, S., Kanwisher, N., & Spelke, E. (2006). Non-symbolic arithmetic in adults and young children. *Cognition, 98*(3), 199–222. https://doi.org/10.1016/j.cognition.2004.09.011.

Barth, H., Starr, A., & Sullivan, J. (2009). Children's mappings of large number words to numerosities. *Cognitive Development, 24*(3), 248–264. https://doi.org/10.1016/j.cogdev.2009.04.001.

Bialystok, E. (1992). Symbolic representation of letters and numbers. *Cognitive Development, 7*(3), 301–316. https://doi.org/10.1016/0885-2014(92)90018-M.

Bonny, J. W., & Lourenco, S. F. (2013). The approximate number system and its relation to early math achievement: Evidence from the preschool years. *Journal of Experimental Child Psychology, 114*(3), 375–388. https://doi.org/10.1016/j.jecp.2012.09.015.

Bussi, M. G. B. (2011). Artefacts and utilization schemes in mathematics teacher education: Place value in early childhood education. *Journal of Mathematics Teacher Education, 14*(2), 93–112. https://doi.org/10.1007/s10857-011-9171-2.

Byrge, L., Smith, L. B., & Mix, K. S. (2014). Beginnings of place value: How preschoolers write three-digit numbers. *Child Development, 85*(2), 437–443. https://doi.org/10.1111/cdev.12162.

Cantrell, L., Boyer, T. W., Cordes, S., & Smith, L. B. (2015). Signal clarity: An account of the variability in infant quantity discrimination tasks. *Developmental Science, 18*(6), 877–893. https://doi.org/10.1111/desc.12283.

Cantrell, L., & Smith, L. (2013). Open questions and a proposal: A critical review of the evidence on infant numerical abilities. *Cognition, 128*(3), 331–352. https://doi.org/10.1016/j.cognition.2013.04.008.

Carey, S. (2001). Cognitive Foundations of Arithmetic: Evolution and Ontogenisis. *Mind & Language, 16*(1), 37–55. https://doi.org/10.1111/1468-0017.00155.

Carey, S. (2010). Beyond fast mapping. *Language Learning and Development: The Official Journal of the Society for Language Development, 6*(3), 184–205. https://doi.org/10.1080/15475441.2010.484379.

Chen, Q., & Li, J. (2014). Association between individual differences in non-symbolic number acuity and math performance: A meta-analysis. *Acta Psychologica, 148*, 163–172. https://doi.org/10.1016/j.actpsy.2014.01.016.

Christie, S., & Gentner, D. (2010). Where hypotheses come from: Learning new relations by structural alignment. *Journal of Cognition and Development, 11*(3), 356–373. https://doi.org/10.1080/15248371003700015.

Clayton, S., Gilmore, C., & Inglis, M. (2015). Dot comparison stimuli are not all alike: The effect of different visual controls on ANS measurement. *Acta Psychologica, 161*, 177–184. https://doi.org/10.1016/J.ACTPSY.2015.09.007.

Cobb, P. G. (1988). Children's initial understandings of ten. *Focus on Learning Problems in Mathematics, 10*(3), 1–28.

Copi, I. M., Menninger, K., & Broneer, P. (1971). Number words and number symbols: A cultural history of numbers. *Philosophy East and West, 21*(1), 97. https://doi.org/10.2307/1397773.

Dehaene, S. (1992). Varieties of numerical abilities. *Cognition*. https://doi.org/10.1016/0010-0277(92)90049-N.

... Dehaene, S. (2007). Symbols and quantities in parietal cortex: Elements of a mathematical theory of number representation and manipulation. *Attention & Performance XXII. Sensori-Motor, 527–574*. https://doi.org/10.1093/acprof:oso/9780199231447.003.0024.

Dehaene, S. (2011). *The number sense: How the mind creates mathematics.* USA: Oxford University Press.

Dehaene, S., & Cohen, L. (1995). Towards an anatomical and functional model of number processing. *Mathematical Cognition*. https://doi.org/10.1155/2014/461647.

Dehaene, S., & Mehler, J. (1992). Cross-linguistic regularities in the frequency of number words. *Cognition, 43*(1), 1–29. https://doi.org/10.1016/0010-0277(92)90030-L.

Fazio, L. K., Bailey, D. H., Thompson, C. A., & Siegler, R. S. (2014). Relations of different types of numerical magnitude representations to each other and to mathematics achievement. *Journal of Experimental Child Psychology, 123*, 53–72. https://doi.org/10.1016/j.jecp.2014.01.013.

Feigenson, L., Dehaene, S., & Spelke, E. (2004). Core systems of number. *Trends in Cognitive Sciences, 8*(7), 307–314. https://doi.org/10.1016/J.TICS.2004.05.002.

Feigenson, L., Libertus, M. E., & Halberda, J. (2013). Links between the intuitive sense of

number and formal mathematics ability. *Child Development Perspectives, 7*(2), 74–79. https://doi.org/10.1111/cdep.12019.
Fisher, C. (1996). Structural limits on verb mapping: The role of analogy in children's interpretations of sentences. *Cognitive Psychology, 31*(1), 41–81. https://doi.org/10.1006/cogp.1996.0012.
Fuson, K. C. (1990). Conceptual structures for multiunit numbers: Implications for learning and reaching multidigit addition, subtraction, and place value. *Cognition and Instruction, 7*(4), 343–403. https://doi.org/10.1207/s1532690xci0704_4.
Fuson, K. C., & Briars, D. J. (1990). Using a base-ten blocks learning/teaching approach for first- and second-grade place-value and multidigit addition and subtraction. *Journal of Research in Mathematic Education, 21*(3), 180–206.
Fuson, K. C., & Kwon, Y. (1991). Chinese-based regular and European irregular systems of number words: The disadvantages for English-speaking children. *Language in Mathematical Education: Research and Practice,* 211–226.
Fuson, K. C., & Kwon, Y. (1992). Korean children's understanding of multidigit addition and subtraction. *Child Development, 63*(2), 491–506. https://doi.org/10.2307/1131494.
Gallistel, C. R., & Gelman, R. (1992). Preverbal and verbal counting and computation. *Cognition, 44*(1–2), 43–74. https://doi.org/10.1016/0010-0277(92)90050-R.
Geary, D. C. (1994). *Children's mathematical development: Research and practical applications.* Washington: American Psychological Association.
Geary, D. C., Bow-Thomas, C. C., Liu, F., & Siegler, R. S. (1996). Development of arithmetical competencies in Chinese and American children: Influence of age, language, and schooling. *Child Development, 67*(5), 2022–2044. https://doi.org/10.1111/j.1467-8624.1996.tb01841.x.
Gentner, D. (1983). Structure-mapping: A theoretical framework for analogy. *Cognitive Science, 7*(2), 155–170. https://doi.org/10.1207/s15516709cog0702_3.
Gentner, D. (2010). Bootstrapping the mind: Analogical processes and symbol systems. *Cognitive Science, 34*(5), 752–775. https://doi.org/10.1111/j.1551-6709.2010.01114.x.
Gentner, D., & Colhoun, J. (2010). Analogical processes in human thinking and learning. In B. Glatzeder, V. Goel, & A. Müller (Eds.). *Towards a theory of thinking* (pp. 35–48). Berlin, Heidelberg: Springer Berlin Heidelberg.
Gentner, D., Levine, S. C., Ping, R., Isaia, A., Dhillon, S., Bradley, C., & Honke, G. (2016). Rapid learning in a children's museum via analogical comparison. *Cognitive Science, 40*(1), 224–240. https://doi.org/10.1111/cogs.12248.
Gervasoni, A., Parish, L., Hadden, T., Turkenburg, K., Bevan, K., Livesey, C., & Croswell, M. (2011). Insights about children's understanding of 2-digit and 3-digit numbers. *Mathematics Traditions and New Practices Proceedings from the Australian Associate*.
Gick, M. L., & Holyoak, K. J. (1983). Schema induction and analogical transfer. *Cognitive Psychology, 15*(1), 1–38. https://doi.org/10.1016/0010-0285(83)90002-6.
Gilmore, C. K., McCarthy, S. E., & Spelke, E. S. (2010). Non-symbolic arithmetic abilities and mathematics achievement in the first year of formal schooling. *Cognition, 115*(3), 394–406. https://doi.org/10.1016/J.COGNITION.2010.02.002.
Ginsburg, H. P., & Russell, R. L. (1981). Social class and racial influences on early mathematical thinking. *Monographs of the Society for Research in Child Development, 46*(6), 1. https://doi.org/10.2307/1165946.
Göbel, S. M., Watson, S. E., Lervåg, A., & Hulme, C. (2014). Children's arithmetic development: It is number knowledge, not the approximate number sense, that counts. *Psychological Science*. https://doi.org/10.1177/0956797613516471.
Goldstone, R. L. (1998). Perceptual learning. *Annual Review of Psychology, 49*, 585–612. https://doi.org/10.1146/annurev.psych.49.1.585.
Goldwater, M. B., Bainbridge, R., & Murphy, G. L. (2016). Learning of role-governed and thematic categories. *Acta Psychologica, 164*(February), 112–126. https://doi.org/10.1016/j.actpsy.2015.10.011.
Grossberg, S., & Repin, D. V. (2003). A neural model of how the brain represents and compares multi-digit numbers: Spatial and categorical processes. *Neural Networks, 16*(8), 1107–1140. https://doi.org/10.1016/S0893-6080(03)00193-X.
Halberda, J., & Feigenson, L. (2008). Developmental change in the acuity of the "Number Sense": The Approximate Number System in 3-, 4-, 5-, and 6-year-olds and adults. *Developmental Psychology, 44*(5), 1457–1465. https://doi.org/10.1037/a0012682.
Halberda, J., Mazzocco, M. M. M., & Feigenson, L. (2008). Individual differences in nonverbal number acuity correlate with maths achievement. *Nature, 455*. https://doi.org/10.1038/nature07246.
Ho, C. S.-H., & Fuson, K. C. (1998). Children's knowledge of teen quantities as tens and ones: Comparisons of Chinese, British, and American kindergartners. *Journal of Educational Psychology, 90*(3), 536–544. https://doi.org/10.1037/0022-0663.90.3.536.
Huang, Y. T., Spelke, E., & Snedeker, J. (2010). When is four far more than three? Children's generalization of newly acquired number words. *Psychological Science, 21*(4), 600–606. https://doi.org/10.1177/0956797610363552.
Huettig, F., Rommers, J., & Meyer, A. S. (2011). Using the visual world paradigm to study language processing: A review and critical evaluation. *Acta Psychologica, 137*(2), 151–171. https://doi.org/10.1016/j.actpsy.2010.11.003.
Hurst, M., Anderson, U., & Cordes, S. (2016). Mapping among number words, numerals, and nonsymbolic quantities in preschoolers. *Journal of Cognition and Development, 1–22*. https://doi.org/10.1080/15248372.2016.1228653.
Hyde, D. C., Khanum, S., & Spelke, E. S. (2014). Brief non-symbolic, approximate number practice enhances subsequent exact symbolic arithmetic in children. *Cognition, 131*(1), 92–107. https://doi.org/10.1016/J.COGNITION.2013.12.007.
Inglis, M., Attridge, N., Batchelor, S., & Gilmore, C. (2011). Non-verbal number acuity correlates with symbolic mathematics achievement: But only in children. *Psychonomic Bulletin & Review, 18*(6), 1222–1229. https://doi.org/10.3758/s13423-011-0154-1.
Jordan, N. C., Kaplan, D., Ramineni, C., & Locuniak, M. N. (2009). Early math matters: Kindergarten number competence and later mathematics outcomes. *Developmental Psychology, 45*(3), 850–867. https://doi.org/10.1037/a0014939.
Kahnt, T., Grueschow, M., Speck, O., & Haynes, J.-D. (2011). Perceptual learning and decision-making in human medial frontal cortex. *Neuron, 70*(3), 549–559. https://doi.org/10.1016/j.neuron.2011.02.054.
Kamii, C. L. (1988). Teaching place value and double-column addition. *Arithmetic Teacher, 35*(6), 48–52.
Kellman, P. J. (2002). Perceptual Learning. *Stevens' Handbook of Experimental Psychology: vol. 3*, (pp. 259–299). Hoboken, NJ, USA: John Wiley & Sons, Inc.
Kellman, P. J., Massey, C. M., & Son, J. Y. (2010). Perceptual learning modules in mathematics: Enhancing students' pattern recognition, structure extraction, and fluency. *Topics in Cognitive Science, 2*(2), 285–305. https://doi.org/10.1111/j.1756-8765.2009.01053.x.
Lachance, J. A., & Mazzocco, M. M. M. (2006). A longitudinal analysis of sex differences in math and spatial skills in primary school age children. *Learning and Individual Differences, 16*(3), 195–216. https://doi.org/10.1016/j.lindif.2005.12.001.
Landy, D., Charlesworth, A., & Ottmar, E. (2016). Categories of large numbers in line estimation. *Cognitive Science, 1–28*. https://doi.org/10.1111/cogs.12342.
Landy, D., & Goldstone, R. L. (2007). How abstract is symbolic thought? *Journal of Experimental Psychology: Learning, Memory, and Cognition, 33*(4), 720–733. https://doi.org/10.1037/0278-7393.33.4.720.
Le Corre, M., & Carey, S. (2007). One, two, three, four, nothing more: An investigation of the conceptual sources of the verbal counting principles. *Cognition, 105*(2), 395–438. https://doi.org/10.1016/j.cognition.2006.10.005.
Leibovich, T., & Henik, A. (2013). Magnitude processing in non-symbolic stimuli. *Frontiers in Psychology, 4*, 375. https://doi.org/10.3389/fpsyg.2013.00375.
Levine, S. C., Suriyakham, L. W., Rowe, M. L., Huttenlocher, J., & Gunderson, E. A. (2010). What counts in the development of young children's number knowledge? *Developmental Psychology, 46*(5), 1309–1319. https://doi.org/10.1037/a0019671.
Libertus, M. E., Feigenson, L., & Halberda, J. (2011). Preschool acuity of the approximate number system correlates with school math ability. *Developmental Science, 14*(6), 1292–1300. https://doi.org/10.1111/j.1467-7687.2011.01080.x.
Libertus, M. E., Feigenson, L., & Halberda, J. (2013). Is approximate number precision a stable predictor of math ability? *Learning and Individual Differences, 25*, 126–133. https://doi.org/10.1016/J.LINDIF.2013.02.001.
Loewenstein, J., Thompson, L., & Gentner, D. (1999). Analogical encoding facilitates knowledge transfer in negotiation. *Psychonomic Bulletin & Review, 6*(4), 586–597. https://doi.org/10.3758/BF03212967.
Lummis, M., & Stevenson, H. W. (1990). Gender differences in beliefs and achievement: A cross-cultural study. *Developmental Psychology, 26*(2), 254–263. https://doi.org/10.1037/0012-1649.26.2.254.
Lyons, I. M., Ansari, D., & Beilock, S. L. (2012). Supplemental material for symbolic estrangement: Evidence against a strong association between numerical symbols and the quantities they represent. *Journal of Experimental Psychology: General*. https://doi.org/10.1037/a0027248.supp.
Lyons, I. M., Bugden, S., Zheng, S., De Jesus, S., & Ansari, D. (2018). Symbolic number skills predict growth in nonsymbolic number skills in kindergarteners. *Developmental Psychology, 54*(3), 440–457. https://doi.org/10.1037/dev0000445.
Lyons, I. M., Nuerk, H.-C., & Ansari, D. (2015). Rethinking the implications of numerical ratio effects for understanding the development of representational precision and numerical processing across formats. *Journal of Experimental Psychology: General, 3*, 1–15. https://doi.org/10.1037/xge0000094.
Mazzocco, M. M. M., Feigenson, L., & Halberda, J. (2011). Impaired acuity of the approximate number system underlies mathematical learning disability (dyscalculia). *Child Development, 82*(4), 1224–1237. https://doi.org/10.1111/j.1467-8624.2011.01608.x.
Miura, I. T., & Okamoto, Y. (1989). Comparisons of U.S. and Japanese first graders' cognitive representation of number and understanding of place value. *Journal of Educational Psychology, 81*(1), 109–114. https://doi.org/10.1037/0022-0663.81.1.109.
Mix, K. S. (2010). Spatial tools for mathematical thought. In K. S. Mix, L. B. Smith, & M. Gasser (Eds.). *The Spatial Foundations of Language and Cognition* (pp. 41–66). Oxford University Press. https://doi.org/10.1093/acprof:oso/9780199553242.003.0003.
Mix, K. S., Prather, R. W., Smith, L., & Stockton, J. D. (2014). Young children's interpretation of multidigit number names: From emerging competence to mastery. *Child Development, 85*(3), 1306–1319. https://doi.org/10.1111/cdev.12197.
Mix, K. S., Smith, L. B., & Crespo, S. (2019). Leveraging relational learning mechanisms to improve place value instruction. In A. Norton, & M. Alibali (Eds.). *Constructing number. Research in mathematics education*. Cham: Springer.
Mix, K. S., Smith, L. B., Stockton, J. D., Cheng, Y.-L., & Barterian, J. A. (2017). Grounding the symbols for place value: Evidence from training and long-term exposure to base-10 models. *Journal of Cognition and Development, 18*(1), 129–151. https://doi.org/10.1080/15248372.2016.1180296.
Mussolin, C., Nys, J., Content, A., Leybaert, J., Barth, H., Kanwisher, N., ... Willburger, E. (2014). Symbolic number abilities predict later approximate number system acuity in preschool children. *PLoS ONE, 9*(3), e91839. https://doi.org/10.1371/journal.pone.0091839.
Namy, L. L., & Gentner, D. (2002). Making a silk purse out of two sow's ears: Young children's use of comparison in category learning. *Journal of Experimental Psychology: General, 131*(1), 5–15. https://doi.org/10.1037/0096-3445.131.1.5.
Odic, D., Le Corre, M., & Halberda, J. (2015). Children's mappings between number words and the approximate number system. *Cognition, 138*, 102–121. https://doi.org/10.1016/j.cognition.2015.01.008.
Odic, D., Libertus, M. E., Feigenson, L., & Halberda, J. (2012). Developmental change in the acuity of approximate number and area representations. *Developmental Psychology, 49*(6), 1103–1112. https://doi.org/10.1037/a0029472.
Park, J., & Brannon, E. M. (2013). Training the approximate number system improves

math proficiency. *Psychological Science, 24*(10), 2013–2019. https://doi.org/10.1177/0956797613482944.
Park, J., & Brannon, E. M. (2014). Improving arithmetic performance with number sense training: An investigation of underlying mechanism. *Cognition, 133*(1), 188–200. https://doi.org/10.1016/j.cognition.2014.06.011.
Piazza, M., Facoetti, A., Trussardi, A. N., Berteletti, I., Conte, S., Lucangeli, D., ... Zorzi, M. (2010). Developmental trajectory of number acuity reveals a severe impairment in developmental dyscalculia. *Cognition, 116*(1), 33–41. https://doi.org/10.1016/J.COGNITION.2010.03.012.
Piazza, M., Izard, V., Pinel, P., Le Bihan, D., & Dehaene, S. (2004). Tuning curves for approximate numerosity in the human intraparietal sulcus. *Neuron, 44*(3), 547–555. https://doi.org/10.1016/J.NEURON.2004.10.014.
Pinhas, M., Donohue, S. E., Woldorff, M. G., & Brannon, E. M. (2014). Electrophysiological evidence for the involvement of the approximate number system in preschoolers' processing of spoken number words. *Journal of Cognitive Neuroscience, 26*(9), 1891–1904. https://doi.org/10.1162/jocn_a_00631.
Prather, R. W. (2012). Connecting neural coding to number cognition: A computational account. *Developmental Science, 15*(4), 589–600. https://doi.org/10.1111/j.1467-7687.2012.01156.x.
Räsänen, P., Salminen, J., Wilson, A. J., Aunio, P., & Dehaene, S. (2009). Computer-assisted intervention for children with low numeracy skills. *Cognitive Development, 24*(4), 450–472. https://doi.org/10.1016/j.cogdev.2009.09.003.
Robinson, N., Abbott, R., Berninger, V., & Busse, J. (1996). The structure of abilities in young, math-precocious children: Gender similarities and differences. *Journal of Educational Psychology, 88*(2), 341–352.
Ross (1986). The development of children's place-value numeration concepts in grades two through five. *Annual Meeting of the American Educational Research Association*. https://doi.org/10.1017/CBO9781107415324.004.
Ross, & Sunflower (1995). *Place-Value: Problem-solving and written assessment using digit-correspondence tasks. The 17th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH*.
Rule, J., Dechter, E., & Tenenbaum, J. B. (2015). Representing and learning a large system of number concepts with latent predicate networks. *CogSci* (pp. 2051–2056). .
Russell, B. (1903). *Principles of mathematics.* Cambridge University Press: Routledge.
Sasanguie, D., Defever, E., Maertens, B., & Reynvoet, B. (2014). The approximate number system is not predictive for symbolic number processing in kindergarteners. *Quarterly Journal of Experimental Psychology, 67*(2), 271–280. https://doi.org/10.1080/17470218.2013.803581.
Saxton, M., & Towse, J. (1998). Linguistic relativity: The case of place value in multi-digit numbers. *Journal of Experimental Child Psychology, 69*(1), 66–79. https://doi.org/10.1006/jecp.1998.2437.
Schneider, M., Beeres, K., Coban, L., Merz, S., Schmidt, S. S., Stricker, J., & De Smedt, B. (2017). Associations of non-symbolic and symbolic numerical magnitude processing with mathematical competence: A meta-analysis. *Developmental Science, 20*(3), https://doi.org/10.1111/DESC.12372.
Smets, K., Sasanguie, D., Szücs, D., & Reynvoet, B. (2015). The effect of different methods to construct non-symbolic stimuli in numerosity estimation and comparison. *Journal of Cognitive Psychology*. https://doi.org/10.1080/20445911.2014.996568.
Sullivan, J., & Barner, D. (2011). Number words, quantifiers, and principles of word learning. *Wiley Interdisciplinary Reviews: Cognitive Science, 2*(6), 639–645. https://doi.org/10.1002/wcs.140.
Sullivan, J., & Barner, D. (2014). Inference and association in children's early numerical estimation. *Child Development, 85*(4), 1740–1755. https://doi.org/10.1111/cdev.12211.
Tanenhaus, M., Spivey-Knowlton, M., Eberhard, K., & Sedivy, J. (1995). Integration of visual and linguistic information in spoken language comprehension. *Science, 268*(5217), 1632–1634. https://doi.org/10.1126/science.7777863.
Thompson, C. A., & Opfer, J. E. (2010). How 15 hundred is like 15 cherries: Effect of progressive alignment on representational changes in numerical cognition. *Child Development, 81*(6), 1768–1786. https://doi.org/10.1111/j.1467-8624.2010.01509.x.
Wagner, J. B., & Johnson, S. C. (2011). An association between understanding cardinality and analog magnitude representations in preschoolers. *Cognition, 119*, 10–22. https://doi.org/10.1016/j.cognition.2010.11.014.
Wai, W., Chan, L., Au, T. K., & Tang, J. (2014). Strategic counting: A novel assessment of place-value understanding. *Learning and Instruction, 29*, 78–94. https://doi.org/10.1016/j.learninstruc.2013.09.001.
Wilson, A. J., Dehaene, S., Dubois, O., & Fayol, M. (2009). Effects of an adaptive game intervention on accessing number sense in low-socioeconomic-status kindergarten children. *Mind, Brain, and Education, 3*(4), 224–234. https://doi.org/10.1111/j.1751-228X.2009.01075.x.
Wynn, K. (1992). Children's acquisition of the number words and the counting system. *Cognitive Psychology, 24*(2), 220–251. https://doi.org/10.1016/0010-0285(92)90008-P.
Zuber, J., Pixner, S., Moeller, K., & Nuerk, H. C. (2009). On the language specificity of basic number processing: Transcoding in a language with inversion and its relation to working memory capacity. *Journal of Experimental Child Psychology, 102*(1), 60–77. https://doi.org/10.1016/j.jecp.2008.04.003.